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Dynamics on Lie groups over local fields and its applications

(국소체 위의 리군에서의 동역학과 그 응용)

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Dynamics on Lie groups over local fields and its applications

A dissertation
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Abstract

Dynamics on Lie groups over local fields and its applications

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We study the structure of reductive algebraic groups over ultra-metric local fields $\mathbb{K}$ and explore some dynamical properties of associated homogeneous spaces. To investigate the behavior of orbits in such spaces, we use the geometry of Euclidean Bruhat-Tits building and relate those dynamical systems to certain shift spaces with countably many alphabets.

First, we prove the exponential mixing property of the action of Cartan subgroups of $\mathbb{K}$-rank 1 algebraic $\mathbb{K}$-groups under certain conditions, which includes the case when the quotient spaces are geometrically finite graphs of groups. This result can be applied to counting the number of closed paths in graphs of groups with an error rates.

We also show when $\mathbb{K}$ is $\mathbb{F}_q(\!(t^{-1})\!)$, on the space of unimodular lattices in $\mathbb{K}^n$ the Birkhoff type Ergodic theorem holds for almost everywhere in every orbit of unipotent groups for maximal singular rays. Using this result, we obtain the finer version of quantitative Khintchine-Groshev type theorem in $\mathbb{K}^n$, which concerns the asymptotic of the number of solutions of certain Diophantine inequalities with weights and directions.

Keywords : Bruhat-Tits buildings, ultra-metric local fields, equidistribution, exponential mixing

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Contents

Chapter 1

Introduction

Let G be a locally compact group, $\Gamma < G$ a discrete subgroup and $H < G$ a closed subgroup. The group H acts on the space $\Gamma\backslash G$ by translations T_h, which is given by $T_h(\Gamma g) = \Gamma gh$. Our main focus in this thesis is the case when G is the $\mathbb{K}$-points of a reductive algebraic group over a non-Archimedean local field $\mathbb{K}$. We are interested in exploring the dynamical properties of the system $(\Gamma\backslash G, \mathcal{B}, \mu, H)$ for the Borel σ-algebra $\mathcal{B}$ of $\Gamma\backslash G$ and an H-invariant measure μ on $\Gamma\backslash G$.

Associated to reductive groups over fields endowed with a non-Archimedean valuation, there are geometric objects called *buildings*, constructed by François Bruhat and Jacques Tits, which allows us to understand the dynamics of these algebraic groups. When the valuation is discrete these Bruhat-Tits buildings are simplicial complexes, and their apartments are affine Euclidean spaces tessellated by simplices with a group of affine isometries as Weyl group. Furthermore, we can view G as a closed subgroup of the automorphism group of its Bruhat-Tits building Δ.

To investigate such dynamical systems concerning the action on buildings, it is useful to consider a shift space with an induced measure with countably many alphabets which is isomorphic to given system. For example, if the $\mathbb{K}$-rank of G is one and a discrete group Γ of G is equal to the associated full group of itself, then such shift spaces are Markov.

Let $\mathbb{G}$ be a semi-simple algebraic $\mathbb{K}$-group of rank 1, $G = \mathbb{G}(\mathbb{K})$ and H a Cartan subgroup of G. In this case G acts on the Bruhat-Tits tree $\mathcal{T}$ of $\mathbb{G}$. Assume that we are given a discrete subgroup Γ of G and a finite equilibrium state μ on $\Gamma\backslash G$. Our first main result is that the dynamical system $(\Gamma\backslash G, \mathcal{B}, \mu, H)$ is mixing with exponential rate under the certain conditions.

Let $\mathcal{T}$ be a locally finite simplicial tree. For a given discrete subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$, let $\pi\colon \mathcal{T} \to \Gamma\backslash\mathcal{T}$ be the natural projection. We call $\Gamma_f = \{g \in \mathrm{Aut}(\mathcal{T}) \,|\, \pi \circ g = \pi\}$ the *associated full subgroup* of Γ. A subgroup $\Gamma < \mathrm{Aut}(\mathcal{T})$ is called *full* if $\Gamma = \Gamma_f$.

Given a potential $\widetilde{F}$ on $E\mathcal{T}$, we have an equilibrium measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ on $\mathcal{GT}$ which is invariant under the action of Γ and the geodesic translation map ϕ. In this case, we have a Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ for which the set of alphabets $\mathcal{S}$ is equal to $E\mathcal{T}$ and the transition probabilities p_{ij} and the invariant distribution π_j are given in terms of the measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ (see chapter 4).

Definition 1.0.1. Let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a full discrete subgroup and $\widetilde{F}\colon E\mathcal{T} \to \mathbb{R}$ be a Γ-invariant real valued function. Let Z_n be the Markov chain with the data $(\mathcal{S}, p_{ij}, \pi_j)$ associated with $(\Gamma\backslash\mathcal{GT}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$. Suppose that there is a function $t\colon \mathcal{S} \to \mathbb{R}_{\geq 0}$ given by $t(s_i) = t_i$, a finite subset $B \subset \mathcal{S}$ and a constant $0 < \rho < 1$ such that for every $s_i \in \mathcal{S} - B$, we have

$$\sum_{s_j} p_{ij} t_j t_i^{-1} \leq \rho. \tag{1.0.2}$$

Then we say that $(\Gamma, \widetilde{F})$ has *property WSG* with (t, B, ρ). If $\widetilde{F}$ is a constant, then we say that Γ has *property WSG* with (t, B, ρ).

The action of Cartan subgroup H of G on $\Gamma\backslash\mathcal{T}$ corresponds to the geodesic translation map ϕ on $\Gamma\backslash\mathcal{GT}$ (see Chapter 3).

Theorem 1.0.3 ([Kw])**.** *Let $\mathcal{T}$ be a locally finite uniform tree and let Γ be a non-elementary full discrete subgroup of $Aut(\mathcal{T})$. Let $\widetilde{F}$ be a potential for Γ such that $(\Gamma, \widetilde{F})$ has property WSG. If $|m_{\Gamma,F}^{\nu^-,\nu^+}| < \infty$ and $L_\Gamma = k\mathbb{Z}$ (see Chapter 3), then for any $f, g \in C_c(\Gamma\backslash\mathcal{G}_o^k\mathcal{T})$, as $n \to \infty$ we have*

$$\left|\int_{\Gamma\backslash\mathcal{G}_o^k\mathcal{T}} (f \circ \phi^{\circ kn}) \cdot g\, dm_{\Gamma,F}^{\nu^-,\nu^+} - \int_{\Gamma\backslash\mathcal{G}_o^k\mathcal{T}} f\, dm_{\Gamma,F}^{\nu^-,\nu^+} \int_{\Gamma\backslash\mathcal{G}_o^k\mathcal{T}} g\, dm_{\Gamma,F}^{\nu^-,\nu^+}\right| = O(\theta^n),$$

for some constant $0 < \theta < 1$. For given $(\Gamma, \widetilde{F})$, the implied constant depends only on f and g.

Proposition 1.0.4. *Let $\Gamma < Aut(\mathcal{T})$ be a geometrically finite discrete subgroup acting on a bi-regular tree $\mathcal{T}_{r+1,s+1}$. Then its associated full group Γ_f satisfies property WSG.*

Using the arguments of [PPS], one can obtain error rate on the number of edge paths of length at most n in $\Gamma\backslash\backslash\mathcal{T}$ with weights $e^{\int_x^{\gamma y} \widetilde{F}}$ using the equidistribution of the skinning measure $d\sigma_{\widetilde{\mathcal{H}}}$ (see Chapter 4). The following corollary gives the precise statement.

Corollary 1.0.5. *Let x be a degree $q^d + 1$ vertex of the Bruhat-Tits tree $\mathcal{T}$ of G and suppose that $(\Gamma, \widetilde{F})$ is a pair of discrete subgroup $\Gamma < G$ and a potential $\widetilde{F}$ for Γ with property WSG. Let $\mathcal{N}_x(n)$ be the weighted counting function of* *the closed path in $\Gamma\backslash\backslash\mathcal{T}$ of length at most n with base point Γx:*

$$\mathcal{N}_x(n) = \sum_{\gamma:\, d_{\mathcal{T}}(\gamma x, x)\leq n} e^{\int_x^{\gamma x} \widetilde{F}}.$$

As $n \to \infty$, we have

$$\mathcal{N}_x(2n) = \frac{e^{2\delta_{\Gamma,F}}\|\nu_x^-\|\|\nu_x^+\||\Gamma_x|}{(e^{2\delta_{\Gamma,F}}-1)\|m_{\Gamma,F}^{\nu^-,\nu^+}\|} e^{2n\delta_{\Gamma,F}} + O(e^{(2\delta_{\Gamma,F}-\kappa)n})$$

for some $\kappa > 0$.

From now on, let $\mathbf{K}$ be the field $\mathbb{F}_q(\!(t^{-1})\!)$ of formal Laurent series in t^{-1} over a finite field $\mathbb{F}_q$ of order q and let $\mathbf{Z}$ be the ring $\mathbb{F}_q[t]$ of polynomials in t over $\mathbb{F}_q$. For $n = l+m$, let $G = SL(n, \mathbf{K})$ and $\Gamma = SL(n, \mathbf{Z})$. Let us denote by $\mathfrak{a}^+$ the set of n-tuples $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{N}^n$ such that

$$\sum_{i=1}^{l} a_i = \sum_{j=1}^{m} a_{l+j}.$$

For $\mathbf{a} = (a_1, \cdots, a_n) \in \mathfrak{a}^+$, let us define

$$g_{\mathbf{a}} = \mathrm{diag}(t^{a_1}, \cdots, t^{a_l}, t^{-a_{l+1}}, \cdots, t^{-a_{l+m}})$$

Our next result is about the action of elements $g_{\mathbf{a}}$ on the space of unimodular $\mathbf{Z}$-lattices G/Γ, for $G = SL(n, \mathbf{K})$ and $\Gamma = SL(n, \mathbf{Z})$. Let us also denote by μ the G-invariant probability measure on $\Gamma\backslash G$. Let

$$u_{\mathbf{y}} = \begin{pmatrix} I_l & \mathbf{y} \\ 0 & I_m \end{pmatrix}$$

and $H = \{u_{\mathbf{y}} \,|\, \mathbf{y} \in \mathrm{Mat}_{l\times m}(\mathbf{K})\}$. When $\mathbf{K}$ and $\mathbf{Z}$ are replaced by $\mathbb{R}$ and $\mathbb{Z}$, the dynamical properties and applications to Diophantine approximations are exhibited in [Gh], [KT], [Gh], [KM] and [KSW]. The following theorem is a field of formal series analogue of the theorem of [KSW].

Theorem 1.0.6 (Pointwise equidistribution)**.** *Let $X = G/\Gamma$. For any given $x \in X$, $\phi \in C_c(X)$ and $\epsilon > 0$, we have*

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi(g_{\mathbf{a}}^n hx) = \int_X \phi d\mu + O(N^{-1/2}(\log N)^{\frac{3}{2}+\epsilon})$$

for almost every $h \in H$.

Furthermore, we obtain the ergodic theorem for certain unbounded functions as appeared in [EMM] and [KSW]. For a unimodular lattice $\Lambda \in X$ and a subgroup $\Delta \leq \Lambda$ with $L = \mathbf{K}\Delta$, let us denote by $d(\Delta)$ the volume of L/Δ and

$$\alpha(\Lambda) = \max\{d(\Delta)^{-1} \colon \Delta \leq \Lambda\}.$$

Let $C_\alpha(X)$ be the space of functions on X satisfying the following properties:

(1) The function $\phi\colon X \to \mathbb{R}$ is continuous except on a μ-null set.

(2) There exists $C > 0$ such that for all $\Lambda \in X$, we have $|\phi(\Lambda)| \leq C\alpha(\Lambda)$.

Theorem 1.0.7. *Let $X = G/\Gamma$ and x_0 be the point $[e] = \Gamma$ in X. For almost every $\mathbf{y} \in Mat_{l\times m}(\mathbf{K})$ and all $f \in C_\alpha(X)$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(g_{\mathbf{a}}^n u_{\mathbf{y}} x_0) = \int_X f d\mu.$$

This thesis is organized as follows. In Chapter 2 we review several definitions and properties of buildings as simplical complexes and the structure of Bruhat-Tits buildings of reductive $\mathbb{K}$-groups. The theory of Bass-Serre for covering theory of graph of groups and the theory of Patterson-Sullivan for constructing the equilibrium state in negatively curved space are presented in Chapter 3. In Chapter 4 we state the definition of property WSG and prove the exponential mixing property of the geodesic translation map in quotient graph of groups which is given by a group Γ with property WSG. Chapter 5 deals with the dynamical system $(SL(n,\mathbb{K})/SL(n,\mathbb{Z}), T_a, \mu)$ and we give the proof of pointwise equidistribution theorem there. Finally, we study the symbolic dynamical system associated to the Cartan actions on higher-rank buildings and discuss some applications in Chapter 6.

Notation

$\mathbb{K}$ a (global or local) field

$\mathbb{G}$ an algebraic group defined over $\mathbb{K}$

$\mathbb{H}$ a closed subgroup of $\mathbb{G}$

$\mathbb{S}$ a maximal $\mathbb{K}$-split torus of $\mathbb{G}$

$\mathbb{T}$ a maximal torus of $\mathbb{G}$

$X^*(\mathbb{G})$ the group of characters

$X_*(\mathbb{G})$ the group of 1-parameter subgroups

$X^*_{\mathbb{K}}(\mathbb{G})$ the group of $\mathbb{K}$-rational characters

$\mathcal{D}(\mathbb{G})$ the derived subgroup $[\mathbb{G}, \mathbb{G}]$

$\mathcal{C}(\mathbb{G})$ the connected center of $\mathbb{G}$

$R(\mathbb{G})$ the radical of $\mathbb{G}$

$R_u(\mathbb{G})$ the unipotent radical of $\mathbb{G}$

$\mathcal{Y}$ a metric tree

$\mathcal{T}$ a simplicial tree

$\mathcal{X}$ a metric building

Δ a simplicial building

G the group of $\mathbb{K}$-points of $\mathbb{G}$

$\mathcal{G}X$ the space of bi-infinite geodesic lines in X

T^1X the space of germs in X

$\mathbf{K}$ the field $\mathbb{F}_q(\!(t^{-1})\!)$ of formal series

$\mathbf{Z}$ the ring $\mathbb{F}_q[t]$ of polynomials

Chapter 2

Buildings and groups

In this chapter, we summarize the group theory that goes along with the theory of buildings, following [AB]. In particular, we will discover a class of groups G for which we can construct an associated building Δ, on which G acts as a group of type-preserving simplicial automorphisms. Also, we review the Bruhat-Tits buildings of reductive algebraic groups over local fields appeared in [BT], which are endowed with a metric with non-positive curvature which makes it look like a Riemannian symmetric space.

2.1 Buildings

2.1.1 Definition and properties

Definition 2.1.1. We summarize here briefly the main properties of buildings.

1. A finite dimensional simplicial complex is called a *chamber complex* if all the maximal cells, called *chambers*, have the same dimension and any two

chambers may be connected by a *gallery*, a sequence of chambers so that consecutive ones have a common codimension 1 face.

2. A *labelled chamber complex* is a chamber complex together with a labelling of the vertices by a set of labels so that the vertices of any maximal simplex are in one to one correspondence with the set of labels.

3. Let (W, S) be a Coxeter group. Associate with it a complex obtained from the partially ordered set whose elements are the cosets $w\langle S'\rangle$ for $w \in W, S' \subset S$, ordered by $A \prec B$ if $B \subset A$. A complex isomorphic to such a complex is called a *Coxeter complex*. This is equivalent to the complex obtained by viewing (W, S) as a reflection group of a Eucldean space and the complex is the complex obtained by the partition of the space by the reflection hyperplanes. A Coxeter complex is a *thin chamber complex*, which is a chamber complex in which every codimension 1 cell belongs to exactly two chambers.

4. Let $\mathcal{H}$ be a set of affine hyperplanes with the following property: every point in the space has an open set which intersects only finitely many hyperplanes from $\mathcal{H}$ and the geometric realization of the corresponding combinatorial complex is an affine space. An *affine Coxeter group* is a group of isometries of an affine space generated by reflections in affine hyperplanes belonging to an invariant set $\mathcal{H}$ of affine hyperplane and this complex is called an *affine Coxeter complex*.

5. A *building* is a complex Δ together with a collection of subcomplexes called *apartments* satisfying the following properties:

   (1) Every apartment is a Coxeter complex.

(2) For each pair of cells $A, B \in \Delta$ there exists an apartment containing it.

(3) If Σ, Σ' are two apartments containing A and B, then there exists an isomorphism $\phi\colon \Sigma \to \Sigma'$ which stabilizes A and B pointwise.

2.1.2 Strongly transitive automorphism groups

Let G be a group of automorphisms of a building Δ. We will say that G acts *strongly transitively* if for any apartment Σ and a chamber $C \in \Sigma$ and an apartment Σ' and $C' \in \Sigma'$ there exists $g \in G$ so that $g\Sigma = \Sigma'$ and $gC = C'$.

Let G be a group acting strongly transitively on a building Δ. Fix an apartment Σ and a chamber $C \in \Sigma$. We define several subgroups of G:

$$B = \{g \in G \,|\, gC = C \text{ pointwise}\},$$
$$N = \{g \in G \,|\, g\Sigma = \Sigma\},$$
$$T = \{g \in N \,|\, g_{|\Sigma} = id_\Sigma\}.$$

For an affine building Δ, we will denote

$$A = \{g \in N \,|\, g \text{ acts on } \Sigma \text{ by translation}\}.$$

We call B a *Borel* subgroup of G, $W = N/T$ the *Weyl group*, and A a *Cartan* *subgroup*. The following diagram summarizes the notation:

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & & \\
B \arrow[dr, dash] & & N \arrow[r, two heads] \arrow[dl, dash] & W = \langle S \rangle \\
 & T & &
\end{tikzcd}$$

Given a building Δ the labelling λ whose restriction ot Σ is the canonical labelling, with S as the set of labels. Then Δ gives rise to a chamber system, consisting of the set $\operatorname{Ch}\Delta$ together with s-adjacency relations, one for each $s \in S$. Since G acts transitively on $\operatorname{Ch}\Delta$ with B as the stabilizer of C, we have a bijection

$$G/B \simeq \operatorname{Ch}\Delta$$

which takes a coset gB to the chamber gC. Given $s \in S$, we need to figure out the s-adjacency relation induced on the set G/B of cosets.

Suppose first that $h \in G$ is an element such that hC is s-adjacent to C. Then $C \cap hC$ is the face $A = C \cap sC$ of C of type $S - \{s\}$. Since h is type-preserving, it must take A to the face of hC of the same type. But A itself is the face of hC of this type, so $hA = A$. Thus hC is s-adjacent to C if and only if h is in the stabilizer P_s of the face A of C of type $S - \{s\}$. Applying the G-action, we conclude that gC is s-adjacent to $g'C$ if and only if $g' = gh$ for some $h \in P_s$.

Lemma 2.1.2. *The group P_s is generated by the elements of B and sT. Moreover, we have $P_s = B \cup BsB$.*

Proof. Given $h \in P_s$, choose an apartment Σ' containing C and hC. By strong transitivity we can find $b \in B$ such that $b\Sigma' = \Sigma$. The chamber bhC of Σ is then s-adjacent to bC, which is equal to C, and hence it is either C or sC. It follows that bh is either in B of in sB. Therefore, we have $h \in B \cap b^{-1}sB$. $\square$

Proposition 2.1.3 ([AB]). *Suppose that G acts strongly transitively on a thick building Δ. Then the following properties hold.*

(1) S consists of elements of order 2 and (W, S) is a Coxeter system.

(2) $B \cup C(S)$ is a subgroup of G for every $s \in S$.

(3) $BW'B$ is a subgroup of G for every special subgroup $W' \subseteq W$.

(4) $G = \coprod_{w \in W} BwB$.

(5) $BsBwB \subseteq BwB \cup BswB$ for every $s \in S$ and $w \in W$.

(6) $BsBwB = BswB$ if $l(sw) \geq l(w)$.

(7) $BsBwB = BwB \cup BswB$ if $l(sw) \leq l(w)$.

(8) For every $s \in S$, we have $sBs^{-1} \nsubseteq B$.

2.1.3 The building associated to a BN-pair

Suppose we are given a quadruple (G, B, N, S), where G is a group; B and N are subgroups which generate G; N normalizes the intersection $T = B \cap N$; and S is a set of generators of the quotient group $W = N/T$. For our convenience, we will denote by $C(w)$ the double coset BwB for $w \in W$ and let $BW'B = \cup_{w \in W'} C(w)$ for every subset W' of W.

Theorem 2.1.4 ([AB]). *It the conditions (5) and (8) hold, then all of the properties (1)-(8) in the Proposition 2.1.3 hold.*

Therefore, our axioms boil down to (5) and (8). We say that a pair of subgroups B and N of a group G is a *BN-pair* if B and N generate G, the intersection $T = B \cap N$ is normal in N, and the quotient $W = N/T$ admits a set of generators S such that the following two conditions hold for all $s \in S$ and $w \in W$:

(BN1) $C(s)C(w) \subseteq C(w) \cup C(sw)$.

(BN2) $sBs^{-1} \nsubseteq B$.

Let us call the above two conditions by *the axiom of BN-pair*. Now assume that G is a group with a BN-pair and that S is as in the definition above. Every subset $S' \subset S$ gives rise to a subgroup $BW'B$ of G for $W' = \langle S' \rangle$. Then the function $S' \mapsto BW'B$ is a poset isomorphism from the set of subsets of S to the set of subgroup of G of the form $BW'B$. We will call a subgroup of G of this form *special*.

Proposition 2.1.5 ([AB]). *The special subgroups of G are precisely the subgroups containing B.*

By a *special coset* in G we mean a coset gP such that P is a special subgroup. Equivalently, a left coset in G is special if and only if it contains a left coset of B. We introduce the poset $\Delta(G, B)$ of special cosets, ordered by the opposite of the inclusion relation. Let Σ be the subcomplex of Δ consisting of the special cosets of the form wP with $w \in W$ and let $\mathcal{A}$ be the set of transforms $g\Sigma$ of Σ by elements of G.

Proposition 2.1.6 ([AB]). *The complex $\Delta = \Delta(G, B)$ is a thick building, and $\mathcal{A}$ is a system of apartments. The action of G is type-preserving and strongly transitive.*

Definition 2.1.7. A map v from $\mathbb{K}$ to $\mathbb{Z} \cup \{\infty\}$ is called *discrete valuation* if it has the following properties:

(a) $v(\mathbb{K}^*) = \mathbb{Z}$

(b) For all $x, y \in \mathbb{K}$, we have $v(x+y) \geq \min\{v(x), v(y)\}$

Let $\mathcal{O} = \{x \in \mathbb{K} \colon v(x) \geq 0\}$. It is a subring of $\mathbb{K}$ and called a *discrete valuation ring*. If we pick an element $\pi \in \mathbb{K}$ with $v(\pi) = 1$, then every element $x \in \mathbb{K}^*$ is uniquely expressible in the form $x = \pi^n u$ with $n \in \mathbb{Z}$ and $u \in \mathcal{O}^*$. The principal ideal $\pi\mathcal{O}$ generated by π can be described in terms of v as $\{x \in \mathbb{K} \colon v(x) > 0\}$. It is a maximal ideal, and hence the quotient ring $\Bbbk = \mathcal{O}/\pi\mathcal{O}$ is a field, called the *residue field* associated to the valuation v.

Example 2.1.8. Consider the group G given by $SL(n, \mathbb{K})$. There is a natural projection p from $SL(n, \mathcal{O})$ to $SL(n, \Bbbk)$. Let us take B by the inverse image of p in $SL(n, \mathcal{O})$ of the upper triangular subgroup of $SL(n, \Bbbk)$. Also, let N be the monomial subgroup of $SL(n, \mathbb{K})$. Then this quadruple (G, B, N, S) satisfies the axiom of BN-pair.

2.2 Bruhat-Tits theory of reductive $\mathbb{K}$-groups

The Bruhat-Tits building of a reductive group $\mathbb{G}$ over a field $\mathbb{K}$ is a polyhedral complex together with a $\mathbb{G}(\mathbb{K})$-action. Its apartments are affine Euclidean spaces equipped with a polyhedral complex. If A is such an apartment, then N acts on A and preserves the polyhedral complex. In this section, we mainly follow [La].

2.2.1 Root system and Coxeter complexes

Let X be a metric space with metric d. For $x \in X$ and $\epsilon > 0$, we let $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$. If A is a real affine space, then we denote by $\mathrm{Aff}(A)$ the group of affine bijections from A to itself. For $x, y \in A$, we denote by (x, y) and $[x, y]$ the open and closed segment in A with end-points x and y, respectively. For an arbitrary subset $U \subseteq A$, we denote by $\overline{U}$ the topological closure of U in A and U° the interior of U.

Let Φ be a root system in a finite-dimensional $\mathbb{R}$-vector space V. A root $a \in \Phi$ is called *divisible*, if $\frac{1}{2}a \in \Phi$. For an arbitrary subset $\Psi \subseteq \Phi$, we let $\Psi^{\mathrm{red}} = \{a \in \Psi \colon \frac{1}{2}a \notin \Psi\}$. If $a, b \in \Phi$, then we let $(a, b) = \{pa+qb \colon p, q \in \mathbb{N}\} \cap \Phi$. A subset $\Psi \subseteq \Phi$ is called *closed*, if $(a, b) \subseteq \Psi$ for all $a, b \in \Psi$. If in addition Psi lies in an open half-space of V, then Ψ is called *positively closed*.

Let $\Psi \subseteq \Phi$ be a positively closed subset. A root $\alpha \in \Psi$ is called *extremal*, if the intersection of $\mathbb{R}^+\alpha$ with any system of generators of the convex cone generated by Ψ is non-empty. An arbitrary total ordering of Ψ^{red} will be called simply *an ordering* of Ψ. An ordering of Ψ is called *good*, if every $\alpha \in \Psi^{\mathrm{red}}$ is an extremal root for the set of all roots which are greater than α.

Let Φ be a root system in V^*. Then Φ defines a Coxeter complex Σ in V such that its faces are the equivalence classes with respect to the following equivalence relation $\sim$. For $x, y \in V$, we have $x \sim y$ if and only if for all $a \in \Phi$, the following condition is valid: $a(x)$ and $a(y)$ have the same sign of are both equal to zero.

There is a canonical bijection between the set of chambers in Σ and the set of bases of Φ. Let $C \in \Sigma$ be a chamber and let $\Delta(C)$ be the basis of Φ

defined by it. There exists a bijection

$$\begin{aligned}\Delta\colon \{F\in\Sigma \mid F\subset \overline{C}\} &\to \mathcal{P}(\Delta(C))\\ F &\mapsto \Delta(F)=\{a\in\Delta(C)\mid a_{|F}>0\}\end{aligned}$$

If $\theta\subseteq\Delta$, then $\Delta^{-1}(\theta)$ will also be denoted by F_θ.

2.2.2 Apartments and buildings of reductive algebraic groups

A $\mathbb{K}$-group $\mathbb{G}$ is called *semi-simple* if $R(\mathbb{G})=1$ and *reductive* if $R_u(\mathbb{G})=1$. A reductive $\mathbb{K}$-group is called *split over* $\mathbb{K}$ if $\mathbb{S}=\mathbb{T}$. In other words, there is a maximal torus which is defined over $\mathbb{K}$ and splits over $\mathbb{K}$. A reductive $\mathbb{K}$-group is called *quasi-split over* $\mathbb{K}$ if there is a Borel subgroup defined over $\mathbb{K}$. In this case, the centralizer of a maximal $\mathbb{K}$-split torus is $\mathbb{K}$-Levi subgroup of a Borel subgroup and therefore equals a maximal torus. Let $\Phi=\Phi(\mathbb{G},\mathbb{S},\mathbb{K})$ be the root system of $\mathbb{G}$ with respect to $\mathbb{S}$. If $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}$, $Ad\colon\mathbb{G}\to GL(\mathfrak{g})$ is the adjoint representation and $a\in\Phi$, then we let

$$\mathfrak{g}_a=\{X\in\mathfrak{g}\mid (Ads)(X)=a(s)X \text{ for all } s\in\mathbb{S}\}.$$

Now we obtain the following characterization of the root groups:

(a) If $a\in\Phi$, then there exists a unique closed, connected, unipotent $\mathbb{K}$-subgroup U_a of $\mathbb{G}$ which is normalized by $Z_{\mathbb{G}}(\mathbb{S})$ and has Lie algebra $\mathfrak{g}_a+\mathfrak{g}_{2a}$ (if $2a\notin\Phi$, then we let $\mathfrak{g}_{2a}=0$).

(b) If $\Psi\subset\Phi$ is positively closed, then there exists a unique closed, connected, unipotent $\mathbb{K}$-subgroup U_Ψ of $\mathbb{G}$ which is normalized by $Z_{\mathbb{G}}(\mathbb{S})$ and has Lie algebra $\sum_{a\in\Psi}\mathfrak{g}_a$.

(c) If $\Psi \subset \Phi$ is positively closed, then the product morphism $\prod_{a\in\Psi^{\mathrm{red}}} U_a \to U_\Psi$ is an isomorphism of $\mathbb{K}$-varieties for each ordering of Ψ.

(d) Let $a, b \in \Phi$ and suppose that a and b are linearly independent. Then (a, b) is positively closed and we have $(U_a, U_b) \subseteq U_{(a,b)}$.

If $\Psi \subseteq \Phi$ is positively closed, then we let $\mathbb{G}_\Psi = \langle U_\Phi, U_{-\Phi}, Z_{\mathbb{G}}(\mathbb{S})\rangle$. For an order on Φ, the groups U_{Φ^+} and U_{Φ^-} will also be denoted by U^+ and U^-, respectively. If Σ denotes the Coxeter complex in $X_*(\mathbb{S})\otimes_{\mathbb{Z}}\mathbb{R}$ defined by Φ and if $C \in \Sigma$ is a chamber, then C defines an order on Φ. The groups U_{Φ^+} and U_{Φ^-} will also be denoted by U_C^+ and U_C^-, respectively.

We have the following decomposition of $G = \mathbb{G}(\mathbb{K})$. Let $C, C' \in \Sigma$ be two chambers. Then (1) $G = U_C^+(\mathbb{K})N_{\mathbb{G}}(\mathbb{S})(\mathbb{K})U_{C'}^+(\mathbb{K})$. (2) For $n, n' \in N_{\mathbb{G}}(\mathbb{S})$, we have $n = n'$ if and only if the double cosets $U_C^+ n U_{C'}^+$ and $U_C^+ n' U_{C'}^+$ are equal. (3) $U_C^+ U_C^- \cap N_{\mathbb{G}}(\mathbb{S}) = \{1\}$.

Lemma 2.2.1 ([Sp98])**.** *Let $a \in \Phi$ and $u \in U_a(\mathbb{K})$. Then $\{m(u)\} \overset{def}{=} N_{\mathbb{G}}(\mathbb{S})(\mathbb{K}) \cap U_{-a}(\mathbb{K})uU_{-a}(\mathbb{K})$ consists of one element. For $u \neq 1$, the element $m(u)$ induces the reflection r_a in $X_*(\mathbb{S})$ and in $X^*(\mathbb{S})$.*

We have a perfect pairing of abelian groups

$$\langle\cdot,\cdot\rangle\colon X_*(\mathbb{S}) \times X^*(\mathbb{S}) \to \mathrm{Hom}(\mathbb{K}^*, \mathbb{K}^*) \simeq \mathbb{Z}$$

which is given by $(\lambda, \chi) \mapsto \langle\lambda, \chi\rangle := \chi \circ \lambda$. By a perfect pairing we mean a bilinear form $V \times W \to F$ for which we have induced isomorphisms $V \to W^*$ and $W \to V^*$. Now let $V_1 = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Then we may identify V_1^* and $X^*(S)\otimes_{\mathbb{Z}}\mathbb{R}$ and extend the perfect pairing $\langle\cdot,\cdot\rangle$ to a perfect pairing $\langle\cdot,\cdot\rangle\colon V_1 \times$

$V_1^* \to \mathbb{R}$. Then, there exists a unique homomorphism $\nu_1 \colon Z \to V_1$ such that for all $g \in Z$ and all $\chi \in X^*_{\mathbb{F}}(Z)$ we have $\langle \chi, \nu_1(g) \rangle = -|\chi(g)|$. This is mainly because that we can identify $X^*_{\mathbb{F}}(Z)$ with a subgroup of finite index in $X^*(S)$. It follows that $H = \mathrm{Ker}(\nu_1)$ is the maximal compact subgroup of Z and is a normal subgroup of N ([La]).

Define $V_0 := \{v \in V_1 \colon a(v) = 0 \text{ for all } a \in \Phi\}$. Let $V = V_1/V_0$ and let $\nu \colon Z \to V$ be the composition $Z \to V_1 \twoheadrightarrow V$. Now let A be an arbitrary affine space under V. Since $\mathrm{Aff}(A) \simeq V \rtimes GL(V)$, we may construct a group homomorphism $\nu' \colon N \to \mathrm{Aff}(A)$ by putting $\nu \colon Z/H \to V$ and $j \colon N/Z \to GL(V)$. More precisely, there exists a unique homomorphism $\nu' \colon N \to \mathrm{Aff}(A)$ for which the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, "\nu"'] & N \arrow[r] \arrow[d, dotted, "\exists\,\nu'"'] & N/Z \arrow[r] \arrow[d, "\bar{\nu}"'] & 0 \\
0 \arrow[r] & V \arrow[r] & \mathrm{Aff}(A) \arrow[r] & GL(V) \arrow[r] & 0
\end{tikzcd}$$

We say the affine space $A = A(G, S)$ together with the group homomorphism $\nu' \colon N \to \mathrm{Aff}(A)$ the *empty apartment* of G with respect to S. The discrete valuation of $\mathbb{F}$ gives A the affine root system. The empty apartment A together with the affine root system and the chambers, faces, etc. is called the *full apartment*.

Definition 2.2.2. For given $a \in \Phi$, $l \in \mathbb{R}$ and $\Omega \subset A$, we define the following several objects.

ϕ_a	the map from $U_a(\mathbb{K})$ to $\mathbb{R} \cup \{\infty\}$ given by $\phi_a(u) = |x_\alpha^{-1}(u)|$
$U_{a,l}$	the subset of $U_a(\mathbb{K})$ given by $\phi_a^{-1}([l, \infty])$
$f_\Omega(a)$	the subset $\inf\{l \in \mathbb{R} \colon a(x) + l \geq 0 \text{ for all } x \in \Omega\}$ of $\mathbb{R}$
$U_{a,\Omega}$	the subset $U_{a, f_\Omega(a)}$ of $U_a(\mathbb{K})$

U_Ω the subgroup $\langle U_{a,\Omega} \colon a \in \Phi \rangle$ of $G(\mathbb{K})$

N_Ω the subset of N given by $\{n \in N \colon \nu'(n)(x) = x \text{ for all } x \in \Omega\}$

P_Ω the subgroup $\langle U_\Omega, N_\Omega \rangle$ of $G(\mathbb{K})$

When $\Omega = \{x\}$, we simply write $f_x(a), U_{a,x}, U_x$ and P_x for $f_\Omega(a), U_{a,\Omega}, U_\Omega$ and P_Ω, respectively. On $G(\mathbb{F}) \times A$ there is an *equivalence relation* defined as follows: $(g, x) \sim (h, y)$ if and only if there is an element $n \in N$ with $y = \nu'(n)(x)$ and $g^{-1}hn \in U_x$. The *Bruhat-Tits building* $\mathcal{B}(G)$ of $G(\mathbb{F})$ is defined by $\mathcal{B}(G) = G(\mathbb{F}) \times A/\sim$. Let d be the metric on A induced by the N/H invariant inner product of V. For any apartment $g \cdot A = A'$, $(x, y) \mapsto d(g^{-1}x, g^{-1}y)$ defines a metric d' on A'. Since N acts by isometries on A, this metric is independent of the choice of g. Now we can extend d to the metric of $\mathcal{B}(G)$ such that the restriction to any apartment A' coincides with the metric d' defined above. It follows that the metric d on $\mathcal{B}(G)$ is $G(\mathbb{F})$-invariant.

Example 2.2.3. Let $\mathbb{G}$ be the algebraic group SL_n and $\mathbb{S}$ the maximal $\mathbb{K}$-split torus of G corresponding to the diagonal matrices. Since $\mathbb{G} = SL_n$ is split, we have $\mathbb{S} = Z_{\mathbb{G}}(\mathbb{S})$. Moreover $Z_b(\mathbb{K})$ is the group of diagonal matrices with entries in $\mathcal{O}^*$. Let Φ be the root system of SL_n with respect to $\mathbb{S}$ and denote by $\xi \colon \mathbb{S} \to \mathbb{G}_m$ the character defined by $\xi_i(\operatorname{diag}(s_1, \cdots, s_n)) = s_i$. Then

$$\Delta = \{\chi_2\chi_1^{-1}, \cdots, \chi_n\chi_{n-1}^{-1}\}$$

is the basis of Φ with respect to the upper unipotent matrices. The root subgroup U_a consists of matrices of the form $1 + e_{ij}$ for elementary matrices e_{ij}.

We may identify $V = X_*(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}^n/\langle(1, \cdots, 1)\rangle$. Let us assume that v is normalized in such a way that $v(\mathbb{K}^*) = \mathbb{Z}$. Then the walls in A are the affine hyperplanes of the form

$$H_{(ij),m} = \{v \in V \mid \langle v, \chi_i \chi_j^{-1} \rangle = m\}$$

for $i, j \in \{1, \cdots, n\}$ with $i \neq j$ and $m \in \mathbb{Z}$. Let W be the Weyl group of Φ. Since $N_{(G)}(\mathbb{S})(\mathbb{K}) \simeq \mathbb{S}(\mathbb{K}) \times W$, there is a natural N action on V.

Chapter 3

Geometry and dynamics of rank 1 buildings

In this chapter, we review several geometric and dynamical properties of groups acting on trees, which includes the theory of Bass-Serre and Patterson-Sullivan. Bass–Serre theory is a group theory that deals with the algebraic structure of groups acting by automorphisms on simplicial trees and can be regarded as one-dimensional version of the orbifold theory, developed in [Se] and [Ba]. Following the paper of [Pa], we analyze the properties of geometrically finite groups acting on trees via the quotient graph of groups obtained from the action. We also give the equilibrium state on the space of geodesic lines in graph of groups via the construction of [PPS], generalizing that of Patterson and Sullivan.

3.1 Groups acting on trees

3.1.1 Bass-Serre theory

For a graph A, we denote by VA the set of vertices of A and by EA the set of *oriented* edges of A. For $e \in EA$, let $\overline{e} \in EA$ be the opposite edge of e and let $\partial_0 e$ and $\partial_1 e$ be the initial vertex and the terminal vertex of e, respectively.

By a *graph of groups* $\mathbf{A} = (A, \mathcal{A})$ we mean a connected graph A together with groups $\mathcal{A}_a$ ($a \in VA$), $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ ($e \in EA$), and monomorphisms $\alpha_e \colon \mathcal{A}_e \to \mathcal{A}_{\partial_1 e}$ ($e \in EA$). An *isomorphism* between two graph of groups $\mathbf{A} = (A, \mathcal{A})$ and $\mathbf{A}' = (A', \mathcal{A}')$ is an isomomorphism $\phi \colon A \to A'$ between two underlying graphs together with the set of isomomorphisms $\phi_a \colon \mathcal{A}_a \to \mathcal{A}'_{\phi(a)}$ and $\phi_e \colon \mathcal{A}_e \to \mathcal{A}'_{\phi(e)}$ satisfying the following property: for each $e \in EA$, there is an element $h_e \in \mathcal{A}'_{\phi(\partial_1 e))}$ such that

$$\phi_{\partial_1 e}(\alpha_e(g)) = h_e \cdot (\alpha'_{\phi(e)}(\phi_e(g))) \cdot h_e^{-1}$$

for all $g \in \mathcal{A}_e$ ([Ba]).

For each $e \in EA$, we put $i(e) = [\mathcal{A}_a \colon \alpha_e \mathcal{A}_e]$. We then call $I(\mathbf{A}) = (A, i)$ the *edge-indexed graph* of $\mathbf{A}$. We will assume that all indices $i(e)$ are finite and define $q(e) = i(\overline{e})/i(e) \in \mathbb{Q}^{\times}_{>0}$ for each $e \in EA$. Let $\pi(A)$ be the path group of A. Since $q(\overline{e}) = q(e)^{-1}$, it follows that $q \colon \pi(A) \to \mathbb{Q}^{\times}_{>0}$ defines a homomorphism.

Given a locally finite tree $\mathcal{T}$, let Γ be a closed subgroup of $\mathrm{Aut}(\mathcal{T})$, which is the group of all isometries of $\mathcal{T}$ acting without inversions. Then the quotient graph $\Gamma\backslash\mathcal{T}$ has a natural structure of graph of groups, which we will denote by $\Gamma\backslash\backslash\mathcal{T}$, as follows. For each $v \in V(\Gamma\backslash\mathcal{T})$ and $e \in E(\Gamma\backslash\mathcal{T})$, choose any corresponding vertex $\widetilde{v} \in V\mathcal{T}$ and edge $\widetilde{e} \in E\mathcal{T}$. Let $\widetilde{\overline{e}} = \overline{\widetilde{e}}$ and fix an element $\gamma_e \in \Gamma$ which satisfies $\gamma_e \widetilde{\partial_1(e)} = \partial_1(\widetilde{e})$. Define $\mathcal{A}_v$ and $\mathcal{A}_e$ be the stabilizer of $\widetilde{v}$ and $\widetilde{e}$ in Γ, respectively, and let $\alpha_e \colon \mathcal{A}_e \to \mathcal{A}_{\partial_1 e}$ be the monomorphism $h \mapsto \gamma_e^{-1} h \gamma_e$. Then the *quotient graph of groups* $\Gamma\backslash\backslash\mathcal{T} = (\Gamma\backslash\mathcal{T}, \mathcal{A})$ does not depend on the choice of $\widetilde{v}, \widetilde{e}$ and γ_e, up to isomorphism of graph of groups. Let $I(\Gamma\backslash\backslash\mathcal{T}) = (\Gamma\backslash\mathcal{T}, i)$ be the edge-indexed graph of $\Gamma\backslash\backslash\mathcal{T}$. Conversely, if $(A, \mathcal{A})$ is a graph of groups and (A, i) is the corresponding edge-indexed graph, then fixing a basepoint $a_0 \in VA$, the universal covering tree $\mathcal{T} = \widetilde{(A, a_0)}$ and the natural projection $\pi \colon \mathcal{T} \to A$ depend only on the edge indexed graph (A, i) ([Se]).

3.1.2 Geometrically finite discrete groups acting on locally finite trees

For a given locally finite simplicial tree $\mathcal{T}$, we define a metric d on $V\mathcal{T}$ so that $d(u, v)$ is the number of edges of the segment between u and v. Let us denote by $S\mathcal{T}$ the space of all bi-infinite geodesics in $\mathcal{T}$, i.e., the set of isometries $\xi \colon \mathbb{Z} \to V\mathcal{T}$. We call an isometry $r \colon \mathbb{N} \cup \{0\} \to V\mathcal{T}$ a *ray*. Let $\widetilde{\phi} \colon \mathcal{GT} \to \mathcal{GT}$ be the *forward geodesic translation* given by $(\widetilde{\phi}\xi)(t) = \xi(t+1)$ and $\widetilde{\iota} \colon \mathcal{GT} \to \mathcal{GT}$ be the *geodesic inversion* given by $(\widetilde{\iota}\xi)(t) = \xi(-t+1)$. Then $\widetilde{\phi}$ and $\widetilde{\iota}$ commute with the action of $\psi \in \mathrm{Aut}(\mathcal{T})$. The group $\mathbb{Z}$ acts on $\mathcal{GT}$ by $n \mapsto \widetilde{\phi}^{\circ n}$.

Let $\widetilde{\pi} \colon \mathcal{GT} \to \Gamma\backslash\mathcal{GT}$ be the natural projection map and let $\phi \colon \Gamma\backslash\mathcal{GT} \to \Gamma\backslash\mathcal{GT}$ be the induced forward geodesic translation given by $\phi \circ \widetilde{\pi} = \widetilde{\pi} \circ \widetilde{\phi}$ and let $\iota \colon \Gamma\backslash\mathcal{GT} \to \Gamma\backslash\mathcal{GT}$ be the induced geodesic inversion defined similarly.

The *Busemann cocycle* is the map $\beta \colon \partial_\infty\mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathbb{Z}$, given by

$$(\omega, x, y) \mapsto \beta_\omega(x, y) = d(x, v) - d(y, v)$$

for some (hence any) $v \in r_x \cap r_y$ where r_x and r_y are the rays starting from x and y, respectively, and converging to ω. The Busemann cocycle satisfies the following properties: $\beta_\omega(x,y)+\beta_\omega(y,z)=\beta(x,z)$, $\beta_\omega(x,y)=-\beta_\omega(y,x)$ and $\beta_{g\cdot\omega}(g\cdot x, g\cdot y)=\beta_\omega(x,y)$ for all $g\in G$, $x,y,z\in\mathcal{T}$ and $\omega\in\partial_\infty\mathcal{T}$.

Given $\xi\in\mathcal{GT}$, let $\xi^+\in\partial_\infty\mathcal{T}$ be the positive end and $\xi^-\in\partial_\infty\mathcal{T}$ be the negative end. A *contracting horosphere* based at ξ^+ is the subset $\mathcal{H}^+_\xi=\{\eta\in\mathcal{GT}\,|\,\eta^+=\xi^+ \text{ and } \beta_{\xi^+}(\xi(0),\eta(0))=0\}$. An *expanding horosphere* based at ξ^- is the subset $\mathcal{H}^-_\xi=\{\eta\in\mathcal{GT}\,|\,\eta^-=\xi^- \text{ and } \beta_{\xi^-}(\xi(0),\eta(0))=0\}$.

Let Γ be a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$. The *limit set* Λ_Γ of Γ is the set of accumulation points of a Γ-orbit in $\mathcal{T}$. By the discreteness of Γ, we have $\Lambda_\Gamma\subseteq\partial_\infty\mathcal{T}$. The *convex hull* $C\Lambda_\Gamma$ of Γ is the smallest convex subset of $\overline{\mathcal{T}}=\mathcal{T}\cup\partial_\infty\mathcal{T}$ containing Λ_Γ.

A point $\omega\in\Lambda_\Gamma$ is called a *conical limit point* if there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of Γ such that for any point $x\in\mathcal{T}$, any ray c which converges to ω and any $n\in\mathbb{N}$, we have $d(\gamma_n x, c)\le C$ for some $C>0$. A point $\omega\in\Lambda_\Gamma$ is called a *horocyclic limit point* if there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of Γ such that for any point $x\in\mathcal{T}$, we have $\lim\limits_{n\to\infty}\beta_\omega(\gamma_n x, x)=\infty$.

Definition 3.1.1. A point $\omega\in\partial_\infty\mathcal{T}$ is called a Γ-*parabolic point* if the stabilizer Γ_ω fixes no point in $\partial_\infty\mathcal{T}$ other than ω and fixes no vertex of $\mathcal{T}$. It is called Γ-*cuspidal* if, further, for any ray with vertex sequence $x_0,x_1,x_2,x_3,\cdots$ toward ω, we have $\Gamma_{x_n}\le\Gamma_\omega$ for any sufficiently large $n>0$.

A point $\omega\in\Lambda_\Gamma$ is called a *bounded parabolic point* if the stabilizer Γ_ω of ω acts properly discontinuously and cocompactly on $\Lambda_\Gamma\backslash\{\omega\}$.

Proposition 3.1.2 ([Pa], Theorem 1.1)**.** *Let Γ be a discrete subgroup of Aut($\mathcal{T}$) and let us denote the minimal Γ-invariant subtree of $\mathcal{T}$ by $\mathcal{T}_{min}$. Then the followings are equivalent:*

(a) Every limit point of Γ is either a conical limit point or a bounded parabolic point.

(b) Every limit point of Γ *is either a horocyclic limit point or a bounded parabolic point.*

(c) The quotient graph of groups $\Gamma\backslash\backslash\mathcal{T}_{min}$ *is a union of a finite graph of finite groups and a finite number of* Γ*-cuspidal rays of groups.*

If the above equivalent conditions hold, then we say the subgroup Γ is *geometrically finite*. If, furthermore, $\Gamma\backslash\backslash\mathcal{T}_{\min}$ is a finite graph of finite groups, then we say Γ is *convex cocompact*. Note that there is a lattice $\Gamma < \mathrm{Aut}(\mathcal{T})$ which is *not* geometrically finite. We also remark here that a discrete subgroup Γ of $\mathrm{Aut}(\mathbb{H}^n)$ is geometrically finite if one of the equivalent conditions (a) and (b) in the above proposition holds, or equivalently, the unit neighborhood of the convex core $\mathcal{C}_\Gamma = \Gamma\backslash C\Lambda_\Gamma$ has finite volume.

A *funnel* F is a subtree of $\mathcal{T}$ with exactly one vertex of degree 1 such that $\mathcal{T} - F$ is connected. Imitating the proof for hyperbolic plane case in [Bo] and using (c) of the above theorem, we have the following proposition about the structure of quotient graph of groups $\Gamma\backslash\backslash\mathcal{T}$ when Γ is geometrically finite.

Proposition 3.1.3. *If* Γ *is geometrically finite, then there are a finite graph of groups* D*, finite* Γ*-cuspidal rays* $C_1, \cdots, C_k$ *and finite funnels* $F_1, \cdots, F_l$ *so that*

(1) $VA = VD \cup VC_1 \cup \cdots \cup VC_k \cup VF_1 \cup \cdots \cup VF_l$.

(2) $|VD \cap VC_j| = |VD \cap VF_m| = 1$ *and* $VC_j \cap VF_m = \phi$.

(3) If $a_{j,0} \in VD \cap VC_j$ *and* $e_{j,1} \in EC_j$ *with* $\partial_1 e_{j,1} = a_{j,0}$*, then* $i(e_{j,1}) = 1$.

Let L_Γ be the subgroup of $\mathbb{Z}$ generated by the translation length $l(\gamma) = \min_{v\in V\mathcal{T}}\{d(v, \gamma\cdot v)\}$ of every element $\gamma \in \Gamma$. We call L_Γ by the *length spectrum* of Γ. If $\mathcal{T}$ has no proper non-empty Γ-invariant subtree and has no vertices of degree 2, then either $L_\Gamma = \mathbb{Z}$ or $L_\Gamma = 2\mathbb{Z}$ ([BP]). If $L_\Gamma = \mathbb{Z}$, then we say the length spectrum of Γ is *not arithmetic*. This property is required to prove that the geodesic translation map is mixing (see Theorem).

3.2 Generalized Patterson-Sullivan theory

The theory of *Patterson-Sullivan densities* was developed by Patterson (for $\mathbb{H}^2$) and Sullivan (for $\mathbb{H}^n, n \geq 3$). It was generalized to groups acting on CAT(-1) spaces by Burger-Mozes ([BM]). See also [CP] for the case of universal covers of finite simplicial graphs. We review in this subsection the Patterson-Sullivan theory together with a non-zero potential $\widetilde{F}$, following [PPS] and [BPP].

3.2.1 Patterson density

A *potential* $\widetilde{F}$ for a discrete group $\Gamma < \mathrm{Aut}(\mathcal{T})$ is a Γ-invariant continuous function $\widetilde{F}\colon E\mathcal{T} \to \mathbb{R}$. We denote by F the induced function $F\colon \Gamma\backslash E\mathcal{T} \to \mathbb{R}$ on the quotient. For all $x, y \in V\mathcal{T}$ with $d(x, y) = n$, we define

$$\int_x^y \widetilde{F} = \sum_{i=0}^{n-1} \widetilde{F}(e_i)$$

where $\xi \in \mathcal{GT}$ is any bi-infinite geodesic with $\xi(0) = x$ and $\xi(n) = y$ and e_i is the edge satisfying $\partial_0 e_i = \xi(i)$ and $\partial_1 e_i = \xi(i + 1)$. This definition does not depend on the choice of $\xi \in \mathcal{GT}$.

Remark 3.3. In general, a potential is a function defined on the unit tangent bundle $T^1\widetilde{M}$ of a negatively curved manifold $\widetilde{M}$ or the space of germs T^1X of a CAT(-1) space X. For simplicial trees $\mathcal{T}$, the unit tangent vector bundle (or the space of germs) is defined as the quotient space $\mathcal{GT}/\sim$ where $\xi \sim \xi'$

if and only if $\xi(0) = \xi'(0)$ and $\xi(1) = \xi'(1)$. Thus, we can canonically identify $\mathcal{GT}/\sim$ with $E\mathcal{T}$.

The *critical exponent* of (Γ, F) is the element $\delta_{\Gamma,F} \in [-\infty, +\infty]$ defined by

$$\delta_{\Gamma,F} = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma, d(x,\gamma y)=n} e^{\int_x^{\gamma y} \widetilde{F}}$$

and the *Poincaré series* of (Γ, F) is the map $Q = Q_{\Gamma,F,x,y} \colon \mathbb{R} \to [0, \infty]$ given by

$$Q \colon s \mapsto \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}-s)}.$$

If $F = 0$, then $\delta_{\Gamma,0}$ and $Q_{\Gamma,0}$ are the usual critical exponent δ_Γ and Poincaré series $\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}$ of Γ, respectively.

Let $\widetilde{F}^+ = \widetilde{F}$ and $\widetilde{F}^- = \widetilde{F} \circ \widetilde{\iota}$. We denote by $F^\pm \colon \Gamma\backslash E\mathcal{T} \to \mathbb{R}$ their induced maps. In the rest of paper, we assume that $\delta = \delta_{\Gamma,F^+} = \delta_{\Gamma,F^-}$ is finite. Generalizing the Busemann cocycle, the authors in [PPS] defined the *Gibbs cocycle* associated with the group Γ and the potential $\widetilde{F}$ as the map $C_F \colon \partial_\infty\mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ given by

$$(\omega, x, y) \mapsto C_{F,\omega}(x, y) = \int_y^v \widetilde{F} - \int_x^v \widetilde{F}$$

for some (hence any) $v \in r_x \cap r_y$. We call $C_{F-\delta}$ a *normalized Gibbs cocycle* associated with Γ and $\widetilde{F}$.

Definition 3.3.1 (Patterson density for $(\Gamma, \widetilde{F})$ ([PPS])). A *Patterson density* of dimension α for the pair $(\Gamma, \widetilde{F})$ is a family of finite nonzero positive Borel measures $\{\mu_x\}_{x\in V\mathcal{T}}$ on $\partial_\infty\mathcal{T}$ such that for every $\gamma \in \Gamma$, for all $x, y \in \mathcal{T}$ and for every $\omega \in \partial_\infty \mathcal{T}$, we have

$$\gamma_* \mu_x = \mu_{\gamma \cdot x} \qquad \text{and} \qquad \frac{d\mu_x}{d\mu_y}(\omega) = e^{-C_{F-\alpha,\omega}(x,y)}.$$

In particular, a $\delta_{\Gamma,F}$-dimensional Patterson density for $(\Gamma, \widetilde{F})$ is called *normalized.*

There is at least one normalized Patterson density for the pair $(\Gamma, \widetilde{F})$. Moreover, if $Q_{\Gamma,F}(\delta) = \infty$, then the normalized Patterson density is unique up to a multiplicative constant ([PPS], [BPP]).

3.3.1 The measure $m_F^{\nu^-,\nu^+}$ on $\Gamma \backslash \mathcal{GT}$

Now let $\{\nu_x^\pm\}$ be the *normalized* Patterson densities for the pairs $(\Gamma, \widetilde{F}^\pm)$, respectively, and let $C^\pm$ be the *normalized* Gibbs cocycle for $(\Gamma, \widetilde{F}^\pm)$. Fixing $o \in \mathcal{T}$, the map $\xi \mapsto (\xi^+, \xi^-, s)$ gives a homeomorphism between $S\mathcal{T}$ and $((\partial_\infty \mathcal{T} \times \partial_\infty \mathcal{T}) \backslash \Delta) \times \mathbb{Z}$, where $\xi(s)$ is the closest point to o on the geodesic line ξ. Let us define the measure $\widetilde{m}_F^{\nu^-,\nu^+}$ on $S\mathcal{T}$ associated to $\{\nu_x^-\}$ and $\{\nu_x^+\}$ by

$$d\widetilde{m}_F^{\nu^-,\nu^+}(\xi) = e^{C^-_{\xi^-}(o,\xi_0)} e^{C^+_{\xi^+}(o,\xi_0)} d\nu_o^-(\xi^-) d\nu_o^+(\xi^+) ds,$$

where ds is the counting measure on $\mathbb{Z}$. It follows from the Γ-invariant conformal property of $\{\nu_x^\pm\}$ that the definition of $\widetilde{m}_F$ is independent of the choice of $o \in \mathcal{T}$ and that $\widetilde{m}_F^{\nu^-,\nu^+}$ is left Γ-invariant. Hence it induces a Radon measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ of the quotient space $\Gamma \backslash \mathcal{GT}$. This definition is also invariant under the geodesic translation map ϕ_Γ on $\Gamma \backslash \mathcal{GT}$. Now we may lift the measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ to $\Gamma \backslash G$ for arbitrary closed subgroup G of $\mathrm{Aut}(\mathcal{T})$ via $M = \mathrm{Stab}_G(\xi)$-invariant extension: for $\widetilde{f} \in C_c(\Gamma\backslash G)$, define

$$m_{\Gamma,F}^{\nu^-,\nu^+}(\widetilde{f}) = \int_{\eta\in\Gamma\backslash\mathcal{GT}} \int_{g\in M} \widetilde{f}(\eta g) dg dm_{\Gamma,F}^{\nu^-,\nu^+}.$$

When F is constant, the normalized Patterson density is equal to the usual δ_Γ-dimensional *Patterson-Sullivan* density and the measure $m_{\Gamma,F}^{\nu^-,\nu^+}$ coincides with the *Bowen-Margulis* measure m_Γ^{BM} given by

$$dm_\Gamma^{\mathrm{BM}} = e^{\delta_\Gamma \beta_{\xi^+}(o,\xi_0)} e^{\delta_\Gamma \beta_{\xi^-}(o,\xi_0)} d\nu_o(\xi^+) d\nu_o(\xi^-) ds.$$

Recall that the length spectrum of Γ is said to be not arithmetic when $L_\Gamma = \mathbb{Z}$ for $\Gamma < \mathrm{Aut}(\mathcal{T})$. The following mixing property of the geodesic flow is due to [Rob] and [BPP].

Theorem 3.3.2. *Let $\Gamma < Aut(\mathcal{T})$ be a discrete subgroup whose length spectrum is not arithmetic. Let $\widetilde{F}$ be a given potential for Γ and let $\nu^\pm$ be the normalized Patterson densities for $(\Gamma, \widetilde{F}^\pm)$. Suppose that $|m_{\Gamma,F}^{\nu^-,\nu^+}| < \infty$. Then, for all compactly supported continuous functions $f, g \colon \Gamma\backslash\mathcal{GT} \to \mathbb{R}$ we have*

$$\int_{\Gamma\backslash\mathcal{GT}} f \cdot (g \circ \phi_\Gamma^{\circ n}) dm_{\Gamma,F}^{\nu^-,\nu^+} \longrightarrow \frac{1}{|m_{\Gamma,F}^{\nu^-,\nu^+}|} \int_{\Gamma\backslash\mathcal{GT}} f dm_{\Gamma,F}^{\nu^-,\nu^+} \int_{\Gamma\backslash\mathcal{GT}} g dm_{\Gamma,F}^{\nu^-,\nu^+}.$$

If $L_\Gamma = k\mathbb{Z}$, then let $S_o^k\mathcal{T}$ be the subset $\{\xi \in \mathcal{GT} \mid d(\xi(0), o) \in k\mathbb{Z}\}$ of $\mathcal{GT}$. This is invariant under Γ and $\phi^{\circ k}$, so we may restrict $m_{\Gamma,F}^{\nu^-,\nu^+}$ to $\Gamma\backslash S_o^k\mathcal{T}$. In this case, the dynamical system $(\Gamma\backslash S_o^k\mathcal{T}, \phi^{\circ k}, m_{\Gamma,F}^{\nu^-,\nu^+})$ is mixing, for any choice of $o \in V\mathcal{T}$ (cf. [BP]).

Chapter 4

Effective equidistribution in infinite graphs of groups

This chapter presents the exponential mixing property of the geodesic translation map on certain kind of graphs of groups, following mainly [Kw].

4.1 Induced Markov chain for the geodesic translation

In this section, we introduce the Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ with countable states which is associated to the dynamical system $(\Gamma\backslash\mathcal{GT}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$. We follow the idea appeared in [BM] and take some notions there with a slight modification. The novelty of this section is that we interpret the exponential mixing properties of the dynamical system $(\Gamma\backslash\mathcal{GT}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ in terms of a countable Markov chain.

4.1.1 Countable Markov shift $X_{(A,i)}$

Let $\mathcal{T}$ be a locally finite tree and $\mathcal{GT}$ be the set of all bi-infinite geodesics in $\mathcal{T}$. Let $\widetilde{\phi}\colon \mathcal{GT} \to \mathcal{GT}$ be the *forward geodesic translation* given by $(\widetilde{\phi}\xi)(t) = \xi(t+1)$ and $\widetilde{\iota}\colon \mathcal{GT} \to \mathcal{GT}$ be the *geodesic inversion* given by $(\widetilde{\iota}\xi)(t) = \xi(-t+1)$. The group $\mathbb{Z}$ acts on $\mathcal{GT}$ by $n \mapsto \widetilde{\phi}^{\circ n}$. Then $\widetilde{\phi}$ and $\widetilde{\iota}$ commute with the action of $\psi \in \mathrm{Aut}(\mathcal{T})$.

Let $\widetilde{\pi}\colon \mathcal{GT} \to \Gamma\backslash\mathcal{GT}$ be the natural projection map and let $\phi\colon \Gamma\backslash\mathcal{GT} \to \Gamma\backslash\mathcal{GT}$ be the induced forward geodesic translation given by $\phi \circ \widetilde{\pi} = \widetilde{\pi} \circ \widetilde{\phi}$ and let $\iota\colon \Gamma\backslash\mathcal{GT} \to \Gamma\backslash\mathcal{GT}$ be the induced geodesic inversion defined similarly. We will assume that Γ is a full discrete subgroup of $\mathrm{Aut}(\mathcal{T})$, i.e., a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$ which is equal to its associated full subgroup $\Gamma_f = \{g \in \mathrm{Aut}(\mathcal{T}) \mid \pi \circ g = \pi\}$ where $\pi\colon \mathcal{T} \to \Gamma\backslash\mathcal{T}$ is the canonical projection.

For a given edge indexed graph (A, i), consider the following subset

$$X_{(A,i)} = \{x = (e_j)_{j\in\mathbb{Z}} \mid \partial_0 e_{j+1} = \partial_1 e_j \text{ and if } e_{j+1} = \overline{e_j}, \text{ then } i^A(e_j) > 1\}$$

of $(EA)^{\mathbb{Z}}$. The family of *cylinders*

$$[e_0, \cdots, e_{n-1}] := \{x \in X_{(A,i)}\colon x_i = e_i, i = 0, \cdots, n-1\}$$

is a basis of open sets for a topology on X. Let $\sigma\colon X_{(A,i)} \to X_{(A,i)}$ be given by $\sigma(x)_i := x_{i+1}$. Then $(\mathcal{GT}, \widetilde{\phi})$ is conjugate to $(X_{\mathcal{T}}, \sigma)$ (Consider $\mathcal{T}$ as $(\mathcal{T}, i^0)$ with $i^0(e) = 1, \forall e \in E\mathcal{T}$). If we denote by (A, i) the edge-indexed graph associated with $\Gamma\backslash\backslash\mathcal{T}$, then we also have a bijection Φ between two spaces $(\Gamma\backslash\mathcal{GT}, \phi)$ and

$(X_{(A,i)}, \sigma)$ so that the following diagram commute.

$$\begin{tikzcd} \Gamma\backslash\mathcal{GT} \arrow[r, "\Phi"] \arrow[d, "\phi"'] & X_{(A,i)} \arrow[d, "\sigma"'] \\ \Gamma\backslash\mathcal{GT} \arrow[r, "\Phi"] & X_{(A,i)} \end{tikzcd} \tag{4.1.1}$$

Indeed, let $\pi\colon \mathcal{T} \to \Gamma\backslash\mathcal{T}$ be the canonical projection and let $\xi\colon \mathbb{Z} \to \mathcal{T}$ be an element of $\mathcal{GT}$. Let ξ_j be the edge satisfying $\partial_0\xi_j = \xi(j)$ and $\partial_1\xi_j = \xi(j+1)$. Define $\widetilde{\Phi}(\xi) = (\pi(\xi_j))_{j\in\mathbb{Z}}$. Then $\widetilde{\Phi}\colon \mathcal{GT} \to X_{(A,i)}$ is Γ-invariant, so it induces the map $\Phi\colon \Gamma\backslash\mathcal{GT} \to X_{(A,i)}$. Given $e_1, e_2 \in E\mathcal{T}$ satisfying $\partial_0 e_1 = \partial_0 e_2$ and $\pi(e_1) = \pi(e_2)$, there is $\gamma \in \mathrm{Aut}(\mathcal{T})$ such that $\gamma e_1 = e_2, \pi\circ\gamma = \pi$, and γ acts trivially on the connected component of $\mathcal{T} - \{e_1, e_2\}$ containing $\partial_0 e_1$. Therefore, Φ is a ϕ-equivariant homeomorphism (cf. [BM]).

4.1.2 Invariant measures on $(X_{(A,i)}, \sigma)$

Let $\mathbf{A} = (A, \mathcal{A})$ be a graph of groups and (A, i) be the corresponding edge-indexed graph. In this subsection, we will get a *Markov measure* λ_F on the dynamical system $(X_{(A,i)}, \sigma)$ corresponding to $m_{\Gamma,F}^{\nu^-,\nu^+}$. In other words, we require

$$\begin{aligned} \sum_{e_j\,:\,\partial_0 e_k = \partial_1 e_j} \lambda_F([e_j, e_k]) &= \lambda_F([e_k]), \\ \sum_{e_k\,:\,\partial_0 e_k = \partial_1 ej} \lambda_F([e_j, e_k]) &= \lambda_F([e_j]) \end{aligned} \tag{4.1.2}$$

and

$$\int_{\Gamma\backslash\mathcal{GT}} f\, dm_{\Gamma,F}^{\nu^-,\nu^+} = \int_{X_{(A,i)}} \Phi^*(f)\, d\lambda_F \tag{4.1.3}$$

where $\Phi^*(f)[(x_i)_{i\in\mathbb{Z}}] = f[\Phi^{-1}((x_i)_{i\in\mathbb{Z}})]$.

Fixing a basepoint $a_0 \in VA$, the universal covering $\mathcal{T} \simeq \widetilde{(\mathbf{A}, a_0)}$ is a locally finite tree. Let $\pi\colon \mathcal{T} \to \Gamma\backslash\mathcal{T}$ be the natural projection. We define the invariant measure λ_F on $(X_{(A,i)}, \sigma)$ as follows. For $e \in E\mathcal{T}$, we denote by

$$\mathcal{O}(e) = \{\omega \in \partial_\infty \mathcal{T} \mid \exists \xi \in \mathcal{G}\mathcal{T} \text{ such that } \xi(0) = \partial_0 e, \xi(1) = \partial_1 e \text{ and } \xi^+ = \omega\},$$

the *shadow* of an edge e. For each cylinder $[e_0, \cdots, e_{n-1}]$, let $\xi_0, \cdots, \xi_{n-1} \in E\mathcal{T}$ be the edges of a path in $\mathcal{T}$ satisfying $\pi(\xi_j) = e_j$ and let

$$\Gamma_{\xi_0,\cdots,\xi_{n-1}} := \mathrm{Stab}_\Gamma(\xi_0) \cap \cdots \cap \mathrm{Stab}_\Gamma(\xi_{n-1}).$$

Let $\nu^\pm$ be the normalized Patterson density for $(\Gamma, \widetilde{F}^\pm)$ constructed in Subsection 3.2.1. Now we define

$$\lambda_F([e_0, \cdots, e_{n-1}]) = \frac{\nu^-_{\partial_0\xi_0}(\mathcal{O}(\overline{\xi_0}))\nu^+_{\partial_1\xi_{n-1}}(\mathcal{O}(\xi_{n-1}))}{|\Gamma_{\xi_0,\cdots,\xi_{n-1}}|} e^{\int_{\partial_0\xi_0}^{\partial_1\xi_{n-1}} \widetilde{F}-\delta}.$$

Since $|\Gamma_{\xi_0,\cdots,\xi_{n-1}}|$ depends only on $e_0, \cdots, e_{n-1}$ and $\nu^\pm$ are Γ-invariant, the measure λ_F is well-defined. Moreover, the equation (4.1.3) holds since

$$\begin{aligned}
&\lambda_F([e_0, \cdots, e_{n-1}]) = \frac{\nu^-_{\partial_0\xi_0}(\mathcal{O}(\overline{\xi_0}))\nu^+_{\partial_1\xi_{n-1}}(\mathcal{O}(\xi_{n-1}))}{|\Gamma_{\xi_0,\cdots,\xi_{n-1}}|} e^{\int_{\partial_0\xi_0}^{\partial_1\xi_{n-1}} \widetilde{F}-\delta} \\
&= \frac{1}{|\Gamma_{\xi_0,\cdots,\xi_{n-1}}|} \int_{\xi^+\in\partial_\infty\mathcal{T}} \mathbb{1}_{\mathcal{O}(\xi_{n-1})} e^{-C_{\xi^+}(\partial_1\xi_{n-1},\partial_0\xi_0)} \int_{\xi^-\in\partial_\infty\mathcal{T}} \mathbb{1}_{\mathcal{O}(\overline{\xi_0})} d\nu^-_{\partial_0\xi_0}(\xi^-) d\nu^+_{\partial_1\xi_{n-1}}(\xi^+) \\
&= \int_{\Gamma\backslash\mathcal{G}\mathcal{T}} \mathbb{1}_{\Phi^{-1}[e_0,\cdots,e_{n-1}]} d\nu^-_{\partial_0\xi_0} d\nu^+_{\partial_0\xi_0} = m^{\nu^-,\nu^+}_{\Gamma,F}(\Phi^{-1}[e_0, \cdots, e_{n-1}]).
\end{aligned}$$

It remains us to prove the Markov property (4.1.2) of λ_F.

Proposition 4.1.4 (Markov property)**.** *Let us define*

$$p_{e_j,e_k} := \frac{\lambda_F([e_j,e_k])}{\lambda_F([e_j])}.$$

Then, the measure λ_F and p_{e_j,e_k} satisfy the following:

$$\sum_{e_j\,:\,\partial_0 e_k=\partial_1 e_j} \lambda_F([e_j])p_{e_j,e_k} = \lambda_F([e_k]) \quad \textit{and} \qquad \sum_{e_k\,:\,\partial_0 e_k=\partial_1 e_j} p_{e_j,e_k} = 1.$$

Proof. Let $\Gamma\backslash\backslash\mathcal{T} = (A,\mathcal{A})$ and $e_k \in EA$ with $\partial_0 e_k = \partial_1 e_j$. If $e_k = \overline{e_j}$, then $|\Gamma_{\xi_k}| = (i^A(e_j)-1)|\Gamma_{\xi_j,\xi_k}|$ and if $e_k \neq \overline{e_j}$, then $|\Gamma_{\xi_k}| = i^A(\overline{e_k})|\Gamma_{\xi_j,\xi_k}|$. In each case, let

$$\rho_{e_j,e_k} = \frac{|\Gamma_{\xi_j}|}{|\Gamma_{\xi_j,\xi_k}|}.$$

Since $\nu^+_{\partial_1\xi_j}(\mathcal{O}_{\xi_k}) = \nu^+_{\partial_1\xi_k}(\mathcal{O}_{\xi_k})e^{\widetilde{F}(\xi_k)-\delta}$ and $\nu^-_{\partial_0\xi_k}(\mathcal{O}_{\overline{\xi_j}}) = \nu^-_{\partial_0\xi_j}(\mathcal{O}_{\overline{\xi_j}})e^{\widetilde{F}(\xi_j)-\delta}$, we have

$$p_{e_j,e_k} = \frac{|\Gamma_{\xi_j}|\nu^+_{\partial_1\xi_j}(\mathcal{O}(\xi_k))}{|\Gamma_{\xi_j,\xi_k}|\nu^+_{\partial_1\xi_j}(\mathcal{O}(\xi_j))} \quad \text{and} \quad \frac{\lambda_F([e_j,e_k])}{\lambda_F([e_k])} = \frac{|\Gamma_{\xi_k}|\nu^-_{\partial_0\xi_k}(\mathcal{O}(\overline{\xi_j}))}{|\Gamma_{\xi_j,\xi_k}|\nu^-_{\partial_0\xi_k}(\mathcal{O}(\overline{\xi_k}))}.$$

By the definition of edge-indexed graph (A,i) (see Subsection **??**) and the finite (even countable) additivity of the normalized Patterson density ν, we have

$$\nu^+_{\partial_1\xi_j}(\mathcal{O}(\xi_j)) = \sum_{e_k\,:\,\partial_0 e_k=\partial_1 e_j} \rho_{e_j,e_k}\nu^+_{\partial_1\xi_j}(\mathcal{O}(\xi_k))$$

and

$$\nu^-_{\partial_0\xi_k}(\mathcal{O}(\xi_k)) = \sum_{e_j\,:\,\partial_1 e_j=\partial_0 e_k} \rho_{e_k,e_j}\nu^-_{\partial_0\xi_k}(\mathcal{O}(\xi_j))$$

which completes the proof. □

The equations in (4.1.2) follow from the above proposition. Therefore, it follows that the measure λ_F is a Markov measure and two dynamical systems

$$(\Gamma\backslash\mathcal{GT}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+}) \text{ and } (X_{(A,i)}, \sigma, \lambda_F)$$

are isomorphic by (4.1.1) and (4.1.3).

4.1.3 Decay of correlation in terms of Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$

We consider a Markov chain Z_n with phase space $\mathcal{S} = \{s_1, s_2, \cdots\}$ and transition probabilities

$$p_{ij} = p_{s_is_j} = P\{Z_{n+1} = s_j \mid Z_n = s_i\}.$$

For a subset $B \subset \mathcal{S}$ of alphabets, let

$$\begin{aligned} f_{ij}^{(n),B} &= P\{Z_1 \notin B \cup s_j, \cdots, Z_{n-1} \notin B \cup s_j, Z_n = s_j \,|Z_0 = s_i\}, \\ p_{ij}^{(n),B} &= P\{Z_1 \notin B, \cdots, Z_{n-1} \notin B, Z_n = s_j \,|Z_0 = s_i\}. \end{aligned} \tag{4.1.5}$$

We also denote by $f_{ij}^{(n)} = f_{ij}^{(n),\phi}$ and $p_{ij}^{(n)} = p_{ij}^{(n),\phi}$. By convention, $f_{ij}^{(0)} = 0$ and $p_{ij}^{(0)} = \delta_{ij}$. Meanwhile, we have the following convolution relation

$$p_{s_is_j}^{(n),B} = \sum_{r=1}^{n} f_{s_is_i}^{(r),B} p_{s_is_j}^{(n-r),B}. \tag{4.1.6}$$

Suppose that the Markov chain Z_n is irreducible, i.e., for any $s_i, s_j \in \mathcal{S}$, there exists $n > 0$ such that $p_{ij}^{(n)} > 0$. We say π_j is the *stationary* (or *invariant*) distribution if it satisfies $\pi_j = \sum_{i \in \mathcal{S}} \pi_i p_{ij}$. If it exists, then we say the Markov chain $(\mathcal{S}, p_{ij})$ is *recurrent* (and otherwise we say *transient*). When the Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ is recurrent, it is called *positive recurrent* if $\sum_{n=1}^{\infty} n f_{jj}^{(n)} < \infty$ (and otherwise called *null recurrent*). When $(\mathcal{S}, p_{ij}, \pi_j)$ is positive recurrent, we have

$$\pi_j = \frac{1}{\sum_{n=1}^{\infty} n f_{jj}^{(n)}}.$$

A *period* of a state $s_i \in \mathcal{S}$ is defined by $k = \gcd\{n \colon p_{ii}^{(n)} > 0\}$. An irreducible Markov chain is called *aperiodic* if for some (and hence every) state $s_i \in \mathcal{S}$, the period is 1. If the positive recurrent Markov chain Z_n is irreducible and aperiodic, then $\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$ and π_j does not depend on the choice of $i \in \mathcal{S}$ ([MT]).

We are interested in the Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ for

$$\mathcal{S} = E(\Gamma \backslash\backslash \mathcal{T}), \quad p_{ij} = p_{s_i s_j} = \frac{\lambda_F([s_i, s_j])}{\lambda_F([s_i])}, \quad \text{and} \quad \pi_j = \lambda_F([s_j]). \tag{4.1.7}$$

If the length spectrum of Γ is not arithmetic, then $(\Gamma \backslash \mathcal{G}\mathcal{T}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ is mixing and hence $(\mathcal{S}, p_{ij}, \pi_j)$ is an irreducible aperiodic Markov chain.

Now we can deduce the exponential decay of correlation from the information of the speed of convergence to stationary distribution.

Proposition 4.1.8. *Suppose there are constants $C_{s_i s_j} > 0$ and $0 < \theta < 1$ such that $|p_{s_i s_j}^{(n)} - \pi_{s_j}| \le C_{s_i s_j} \theta^n$. Then*

$$|Cov(f, g)| = \left| \int (f \circ \sigma^{\circ n}) \cdot g d\lambda_F - \int f d\lambda_F \int g d\lambda_F \right| \le C(f, g) \theta^n.$$

Proof. Since every measurable set can be approximated by finite disjoint unions of cylinder sets, we only need to show that $|\lambda_F([\underline{a}] \cap \sigma^{-n}[\underline{b}]) - \lambda_F([\underline{a}])\lambda_F([\underline{b}])|$ converges to 0 exponentially, for all cylinder sets $[\underline{a}] = [a_0, \cdots, a_{k-1}]$ and $[\underline{b}] = [b_0, \cdots, b_{l-1}]$. Now,

$$\begin{aligned}
\left|\mathrm{Cov}(\mathbb{1}_{[\underline{a}]}, \mathbb{1}_{[\underline{b}]})\right| &= \left|\lambda_F([\underline{a}] \cap \sigma^{-n}[\underline{b}]) - \lambda_F([\underline{a}])\lambda_F([\underline{b}])\right| \\
&= \left|\lambda_F\left(\bigcup_{\underline{c} \in W_{n-k}} [\underline{a}, \underline{c}, \underline{b}]\right) - \lambda_F([\underline{a}])\lambda_F([\underline{b}])\right| \\
&= \left|\lambda_F([\underline{a}])\left(\sum_{\underline{c} \in W_{n-k}} p_{a_{k-1}c_0} \cdots p_{c_{n-k-1}b_0} \cdot \frac{\lambda_F([\underline{b}])}{\pi_{b_0}}\right) - \lambda_F([\underline{a}])\lambda_F([\underline{b}])\right| \\
&= \lambda_F([\underline{a}])\lambda_F([\underline{b}]) \left|\frac{p^{(n-k)}_{a_{k-1}b_0} - \pi_{b_0}}{\pi_{b_0}}\right| \leq \|\mathbb{1}_{[\underline{a}]}\| \|\mathbb{1}_{[\underline{b}]}\| \frac{C_{a_{k-1}b_0}}{\pi_{b_0}\theta^k} \theta^n.
\end{aligned}$$

This completes the proof. □

4.2 Exponential mixing and its applications

Let $\mathcal{T}$ be a locally finite tree. Let $(\Gamma, \widetilde{F})$ be a pair of full discrete subgroup $\Gamma <$ Aut$(\mathcal{T})$ and a potential $\widetilde{F}$ for Γ with $|m^{\nu^-,\nu^+}_{\Gamma,F}| < \infty$. Assume that $L_\Gamma = \mathbb{Z}/k\mathbb{Z}$ and let $S^k_o\mathcal{T}$ be a subset $\{\xi \in \mathcal{GT} \,|\, d(\xi_0, o) \in k\mathbb{Z}\}$ of $\mathcal{GT}$ for $o \in V\mathcal{T}$. Our goal in this section is to prove the exponential decay of the correlation function

$$|\mathrm{Cov}(f, g)| = \left|\int_{\Gamma\backslash S^k_o\mathcal{T}} (f \circ \sigma^{\circ n}) \cdot g dm^{\nu^-,\nu^+}_{\Gamma,F} - \int_{\Gamma\backslash S^k_o\mathcal{T}} f dm^{\nu^-,\nu^+}_{\Gamma,F} \int_{\Gamma\backslash S^k_o\mathcal{T}} g dm^{\nu^-,\nu^+}_{\Gamma,F}\right|$$

for $f, g \in C_c(\Gamma\backslash S^k_o\mathcal{T})$ when $\Gamma <$ Aut$(\mathcal{T})$ has WSG property (see Definition 4.2.2). By Proposition 4.1.8, it suffices to show that $|p^{(n)}_{s_is_j} - \pi_{s_j}| \leq C_{s_is_j}\theta^n$ for some $C_{s_i s_j} > 0$ and $0 < \theta < 1$. Inspired by L.-S. Young's observation about the relation between the speed of convergence to equilibrium states and the recurrent time to a compact set ([Y]), we investigate the speed of decay of the probability $p^{(n),B}_{s_i s_j}$ for a finite set $B \subset E(\Gamma\backslash\mathcal{T})$ (see (4.1.5)).

4.2.1 Exponential convergence to the stationary distribution

Definition 4.2.1 (Property WSG)**.** Let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a full discrete subgroup and $\widetilde{F} \colon E\mathcal{T} \to \mathbb{R}$ be a potential for Γ such that $|m^{\nu^-,\nu^+}_{\Gamma,F}| < \infty$. Let Z_n be the Markov chain with the data $(\mathcal{S}, p_{ij}, \pi_j)$ associated with $(\Gamma\backslash\mathcal{GT}, \phi, m^{\nu^-,\nu^+}_{\Gamma,F})$ (see (4.1.7)). Suppose that there is a function $t \colon \mathcal{S} \to \mathbb{R}_{\geq 0}$ given by $t(s_i) = t_i$, a finite subset $B \subset \mathcal{S}$ and a constant $0 < \rho < 1$ such that for $s_i \in \mathcal{S} - B$, we have

$$\sum_{s_j} p_{ij} t_j t_i^{-1} \leq \rho. \tag{4.2.2}$$

Then we say $(\Gamma, \widetilde{F})$ has property *WSG* (*weighted spectral gap*) with (t, B, ρ).

Lemma 4.2.3. *Suppose* $(\Gamma, \widetilde{F})$ *has property WSG with* $t \colon \mathcal{S} \to \mathbb{R}_{\geq 0}$*,* $B \subset \mathcal{S}$ *and* $0 < \rho < 1$*. Then, we obtain* $p^{(n),B}_{ij} \leq t_i t_j^{-1} \rho^n$*. In particular,*

$$p^{(n),B}_{i,B} = P\{Z_1 \notin B, \cdots, Z_{n-1} \notin B, Z_n \in B \,|\, Z_0 = s_i\} \leq M t_i \rho^n$$

for $M = \max\{t_j^{-1} \,|\, s_j \in B\}$.

Proof. Let $t_i = t(s_i)$ and $t_j = t(s_j)$. Then

$$\begin{aligned}
p_{ij}^{(n),B} &= \sum_{r_k \in \mathcal{S}-B} p_{s_i r_1} p_{r_1 r_2} \cdots p_{r_{n-1} s_j} = \frac{t_i}{t_j} \sum_{r_k \in \mathcal{S}-B} \left(p_{s_i r_1} \frac{t(r_1)}{t_i} \right) \cdots \left(p_{r_{n-1} s_j} \frac{t_j}{t(r_{n-1})} \right) \\
&\leq \frac{t_i}{t_j} \left(\sum_{r_1} p_{s_i r_1} \frac{t(r_1)}{t_i} \right) \left(\max_{r_1} \sum_{r_2} p_{r_1 r_2} \frac{t(r_2)}{t(r_1)} \right) \cdots \left(\max_{r_{n-1}} p_{r_{n-1} j} \frac{t_j}{t(r_{n-1})} \right) \\
&\leq t_i t_j^{-1} \rho^n.
\end{aligned}$$

The second statement follows directly from this inequality. □

The following two propositions are main ingredients of our proof of exponential mixing property.

Proposition 4.2.4. *Let $(\mathcal{S}, p_{ij})$ be an irreducible aperiodic countable Markov chain. If there exists a state $b_0 \in \mathcal{S}$ which satisfies $p_{s_i s_j}^{(n),\{b_0\}} \leq C_{s_i,s_j} \theta^n$ for some $0 < \theta < 1$, then it converges to the stationary distribution exponentially, i.e., $|p_{s_i s_j}^{(n)} - \pi_{s_j}| < \widehat{C}_{s_i,s_j} \widehat{\theta}^n$ for some $0 < \widehat{\theta} < 1$ with the constants $\widehat{C}_{s_i,s_j}$ depending only on s_i and s_j.*

Proof. Let $F_{s_i s_j}(z) = \sum_{n=0}^{\infty} f_{ij}^{(n)} z^n$ and $P_{s_i s_j}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n$ be the generating functions. By the assumption, we have $f_{s_i b_0}^{(n)} = p_{s_i b_0}^{(n),\{b_0\}} \leq C_{s_i} \theta^n$. Thus, power series $F_{s_i b_0}(z)$ is analytic in the disk $C_{\theta^{-1}} = \{z \in \mathbb{C} \mid |z| < \theta^{-1}\}$. Moreover, from the convolution relation (4.1.6), the equality

$$P_{b_0 b_0}(z) = \frac{1}{1 - F_{b_0 b_0}(z)}$$

holds and $P_{b_0 b_0}(z)$ has a unique simple pole at the point $z = 1$ in the disk C_R

for some $1 < R \leq \theta^{-1}$. Since

$$\lim_{z \to 1} \frac{z-1}{1 - F_{b_0 b_0}(z)} = \frac{1}{-\sum_{n=1}^{\infty} n f_{b_0 b_0}^{(n)}} = -\pi_{b_0},$$

it follows that $P_{b_0 b_0}(z) - \frac{\pi_{b_0}}{1-z} = \sum\limits_{n=0}^{\infty} \left(p_{b_0 b_0}^{(n)} - \pi_{b_0} \right) z^n$ is analytic for $|z| < R$ and hence there is a constant $c > 0$ such that

$$|p_{b_0 b_0}^{(n)} - \pi_{b_0}| < c\theta_1^n$$

for some $\theta_1 \leq R^{-1} < 1$.

Now using the equality

$$p_{s_i b_0}^{(n)} = f_{s_i b_0}^{(n)} + \sum_{r=1}^{n-1} f_{s_i, b_0}^{(r)} p_{b_0 b_0}^{(n-r)}$$

and the triangle inequality, we have

$$|p_{s_i b_0}^{(n)} - \pi_0| \leq \left| \sum_{r=1}^{n} f_{s_i b_0}^{(r)} (p_{b_0 b_0}^{(n-r)} - \pi_{b_0}) \right| + \pi_{b_0} \sum_{r=n+1}^{\infty} f_{s_i b_0}^{(r)}.$$

This implies the existence of constant $C'_{s_i} > 0$ for which

$$|p_{s_i b_0}^{(n)} - \pi_{b_0}| < C'_{s_i} \theta_2^n$$

with some $R^{-1} < \theta_2 < 1$.

Finally, since the Markov chain $(\mathcal{S}, p_{ij}, \pi_j)$ is irreducible and aperiodic, we have $\pi_{s_j} = \pi_{b_0} \sum_{n=0}^{\infty} p_{b_0,s_j}^{(n),\{b_0\}}$ and

$$p_{s_i s_j}^{(n)} = p_{s_i,s_j}^{(n),\{b_0\}} + \sum_{r=0}^{n-1} p_{s_i b_0}^{(r)} p_{b_0 s_j}^{(n-r),\{b_0\}}.$$

Hence, we obtain

$$|p_{s_i s_j}^{(n)} - \pi_{s_j}| \le p_{s_i s_j}^{(n),\{b_0\}} + \left| \sum_{r=1}^{n-1} (p_{s_i b_0}^{(r)} - \pi_{b_0}) p_{b_0 s_j}^{(n-r),\{b_0\}} \right| + \pi_{b_0} \sum_{r=n}^{\infty} p_{b_0,s_j}^{(r),\{b_0\}}$$

which implies that there is a constant $C(s_i, s_j)$ such that

$$|p_{s_i s_j}^{(n)} - \pi_{s_j}| < \widehat{C}_{s_i s_j} \widehat{\theta}^n$$

for some $\theta_2 < \widehat{\theta} < 1$. This completes the proof. $\square$

Proposition 4.2.5. *Let $C_{s_i s_j}$ be a positive constant depending only on s_i and s_j. If there is a finite subset $B_N = \{b_0, b_1, \cdots, b_N\} \subset \mathcal{S}$ of alphabets such that $p_{s_i s_j}^{(n),B_N} \le C_{s_i s_j} \tau^n$ for some $0 < \tau < 1$, then the Markov chain $(\mathcal{S}, p_{ij})$ converges to the stationary distribution exponentially.*

Proof. Since

$$p_{s_i s_j}^{(n),B_{N-1}} = p_{s_i s_j}^{(n),B_N} + \sum_{r=1}^{n-1} f_{s_i b_N}^{(r),B_{N-1}} p_{b_N s_j}^{(n-r),B_{N-1}}, \tag{4.2.6}$$

setting $s_i = b_N$, it follows that

$$P^{B_{N-1}}_{b_N s_j}(z) = \frac{P^{B_N}_{b_N s_j}(z)}{1 - F^{B_{N-1}}_{b_N b_N}(z)}.$$

Thus, $P^{B_{N-1}}_{b_N s_j}(z)$ is analytic in the disk C_R for some $R > 1$ and we obtain the estimate $p^{(n),B_{N-1}}_{b_N s_j} < c_{s_j}\widetilde{\tau}^n$ for some $\widetilde{\tau} < 1$. Moreover, $f^{(n),B_{N-1}}_{s_i b_N} = p^{(n),B_N}_{s_i b_N} \leq C_{s_i b_N}\tau^n$. Now from (4.2.6), we get

$$p^{(n),B_{N-1}}_{s_i s_j} \leq \widehat{C}_{s_i s_j}\widehat{\tau}^n.$$

for some $\widehat{C}_{s_i s_j} > 0$ and $\widetilde{\tau} < \widehat{\tau} < 1$ Using induction for the sets $B_{N-1}, B_{N-2}, \cdots$, it finally gives us

$$p^{(n),\{b_0\}}_{s_i s_j} \leq C'_{s_i s_j}\theta^n$$

for some $\tau < \theta < 1$. Now we can apply the previous Lemma. □

Now we state and prove the main theorem of this section.

Theorem 4.2.7. *Let $\mathcal{T}$ be a locally finite tree. Suppose that $(\Gamma, \widetilde{F})$ is a pair of a full discrete subgroup $\Gamma < Aut(\mathcal{T})$ and a potential $\widetilde{F}$ for Γ with $|m^{\nu^-,\nu^+}_{\Gamma,F}| < \infty$. Let us fix a vertex $o \in V\mathcal{T}$. If $L_\Gamma = k\mathbb{Z}$ and $(\Gamma, \widetilde{F})$ has property WSG, then for any $f, g \in C_c(\Gamma\backslash S^k_o\mathcal{T})$, we have*

$$\left| \int_{\Gamma\backslash S^k_o\mathcal{T}} (f \circ \phi^{\circ kn}) \cdot g \, dm^{\nu^-,\nu^+}_{\Gamma,F} - \int_{\Gamma\backslash S^k_o\mathcal{T}} f \, dm^{\nu^-,\nu^+}_{\Gamma,F} \int_{\Gamma\backslash S^k_o\mathcal{T}} g \, dm^{\nu^-,\nu^+}_{\Gamma,F} \right| = O(\theta^n)$$

for some constant $0 < \theta < 1$. The implied constant depends only on f and g.

Proof. The dynamical system $(\Gamma\backslash\mathcal{GT}, \phi, m_{\Gamma,F}^{\nu^-,\nu^+})$ is isomorphic to $(X_{(A,i)}, \sigma, \lambda_F)$. Let Z_n be the Markov chain with the data $(\mathcal{S}, p_{ij}, \pi_j)$, where $\mathcal{S} = E(\Gamma\backslash\mathcal{GT})$, $p_{ij} = \frac{\lambda_F([s_i,s_j])}{\lambda_F([s_i])}$ and $\pi_j = \lambda_F([s_j])$. If $L_\Gamma = k\mathbb{Z}$, then $(\Gamma\backslash S_o^k\mathcal{T}, \phi^{\circ k}, m_{\Gamma,F}^{\nu^-,\nu^+})$ is mixing. Therefore, if we let Z_{kn} be the k-step Markov chain on the equivalence of periodic classes in $(\mathcal{S}, p_{ij}, \pi_j)$, then it is irreducible and aperiodic. Since $(\Gamma, \widetilde{F})$ has property WSG, by Lemma 5.2.4 there is a finite subset B of $\mathcal{S}$ such that $p_{ij}^{(kn),B} \leq t_i t_j^{-1}\rho^{kn}$ for some $0 < \rho < 1$. It follows that there exists $C_{ij}, \theta > 0$ such that $|p_{ij}^{(kn)} - \pi_j| < C_{ij}\theta^{kn}$ by Proposition 4.2.5. Now Proposition 4.1.8 completes the proof. □

4.2.2 Examples of $\Gamma < \mathbf{Aut}(\mathcal{T})$ with property WSG

First, we state and prove the fact about the largeness of the critical exponent δ_Γ of Γ when $\Gamma\backslash\backslash\mathcal{T}$ has a cuspidal ray. This is due to the finiteness of the normalized Patterson density.

Proposition 4.2.8. *Let $\mathcal{T}$ be a locally finite tree. Let $(\Gamma, \widetilde{F})$ be a pair of non-elementary full discrete subgroup $\Gamma < Aut(\mathcal{T})$ and a potential $\widetilde{F}$ for Γ with $\left|m_{\Gamma,F}^{\nu^-,\nu^+}\right| < \infty$. If $\Gamma\backslash\backslash\mathcal{T}$ has at least one cuspidal ray (see Definition 3.1.1) with edges $e_1, e_2, e_3, \cdots$ and indices $r_1, r_2, r_3, \cdots$, then*

$$\delta_{\Gamma,F} \geq \frac{1}{2}\log \overline{\lim}_{n\to\infty}(c_n^{1/n})$$

where $c_n = (r_n - 1)r_{n-1}r_{n-2}\cdots r_1 e^{F(e_1)+F(\overline{e_1})+\cdots F(e_n)+F(\overline{e_n})}$. Furthermore, if the tree $\mathcal{T}$ is regular or bi-regular and F is constant, then the inequality is strict.

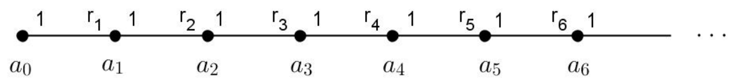

Figure 4.1: Cuspidal ray with index $r_1, r_2, r_3, \cdots$

Proof. Let e_i be the edge with $\partial_0 e_i = a_{i-1}$ and $\partial_1 e_i = a_i$. Let $\nu^{\pm}$ be the Patterson density for $(\Gamma, \widetilde{F}^{\pm})$ defined in Subsection 3.2.1 and let $x = \nu^+_{a_0}(\mathcal{O}(\overline{e_1}))$. Since Γ is non-elementary, ν^+ has no atoms (cf. [Bo]) and it follows that $x \neq 0$. Let $G\colon E\mathcal{T} \to \mathbb{R}$ be a Γ-invariant function given by $G(e) = F(e) + F(\overline{e})$. Because of the countably additivity and conformal property of ν^+, we have

$$\begin{aligned}\nu^+_{a_0}(\mathcal{O}(e_1)) &= (r_1-1)\nu^+_{a_1}(\mathcal{O}(\overline{e_1}))e^{F(e_1)-\delta} + (r_2-1)r_1\nu^+_{a_2}(\mathcal{O}(\overline{e_1}))e^{F(e_1)+F(e_2)-2\delta} + \cdots \\ &= (r_1-1)xe^{G(e_1)-2\delta} + (r_2-1)r_1xe^{G(e_1)+G(e_2)-4\delta} + \cdots \\ &= \sum_{n=1}^{\infty} c_n x e^{-2n\delta} < \infty\end{aligned}$$

which implies $\delta_{\Gamma,F} \geq \frac{1}{2}\log \overline{\lim}_{n\to\infty}(c_n^{1/n})$. If we assume further that $\mathcal{T}$ is a $(r+1, s+1)$ bi-regular tree $\mathcal{T}_{r+1,s+1}$ and F is constant, then

$$\nu^+_{a_0}(\mathcal{O}(e_1)) = \left[(r-1)xe^{2(F-\delta)} + (s-1)rxe^{4(F-\delta)}\right]\sum_{n=1}^{\infty}(rs)^n e^{4n(F-\delta)},$$

so we obtain $\delta_{\Gamma,F} > F + \frac{1}{2}\log\sqrt{rs}$. $\square$

In the rest of this subsection, we assume that the given potential $\widetilde{F}\colon E\mathcal{T} \to \mathbb{R}$ for $\Gamma < \mathrm{Aut}(\mathcal{T})$ is constant. We give some examples of groups Γ which have property WSG with these potentials.

Example 4.2.9. Let $\mathcal{T}_{r+1,s+1}$ be a $(r+1, s+1)$ bi-regular tree and let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a geometrically finite full discrete subgroup with critical exponent δ_Γ for the zero potential. Then we have a decomposition $V(\Gamma\backslash\mathcal{T}) = VD \cup VC_1 \cup \cdots \cup VC_k \cup VF_1 \cup \cdots \cup VF_l$ of the set of vertices as in Proposition (3.1.3) and there is a finite set $B \subseteq E(\Gamma\backslash\mathcal{T})$ such that $E(\Gamma\backslash\mathcal{T}) = EB \cup EC_1 \cup \cdots \cup EC_k \cup EF_1 \cup \cdots EF_l$.

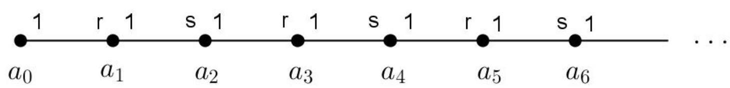

Figure 4.2: Cuspidal ray of $\mathcal{T}_{r+1,s+1}$

Let $\widetilde{F}$ be a given constant potential. Note that $F - \delta_{\Gamma,F} = -\delta_\Gamma$. For a cuspidal ray with the sequence of vertices $a_0, a_1, a_2, \cdots$, if we let e_i be the oriented edge with $\partial_0 e_i = a_{i-1}$ and $\partial_1 e_i = a_i$, then

$$p_{e_{2i-1}e_{2i}} = \frac{(s-1)re^{-2\delta_\Gamma} + (r-1)rse^{-4\delta_\Gamma}}{(r-1) + (s-1)re^{-2\delta_\Gamma}} < 1\,,$$

$$p_{e_{2i}e_{2i+1}} = \frac{(r-1)se^{-2\delta_\Gamma} + (s-1)rse^{-4\delta_\Gamma}}{(s-1) + (r-1)se^{-2\delta_\Gamma}} < 1\,,$$

$$p_{e_i\overline{e_i}} = 1 - p_{e_ie_{i+1}}\,,$$

and

$$p_{\overline{e_{i+1}}\,\overline{e_i}} = 1.$$

Let $p = p_{e_{2i-1}e_{2i}}$ and $q = p_{e_{2i}e_{2i+1}}$. Define $t\colon E\mathcal{T} \to \mathbb{R}_{\geq 0}$ as follows: Choose

$1 < R < \sqrt[4]{pq}$ and t_1 sufficiently large. Let $t(\overline{e}_i) = R^i$ and $t(e_i) = t_i$ with

$$t_{2i} = \frac{1}{p^i q^{i-1} R^{2i-1}} \left(t_1 - (1-p)R^2 \sum_{k=1}^{i} (pqR^4)^k - (1-q)pR^4 \sum_{k=1}^{i-1} (pqR^4)^k \right)$$
$$t_{2i+1} = \frac{1}{p^i q^i R^{2i}} \left(t_1 - (1-p)R^2 \sum_{k=1}^{i} (pqR^4)^k - (1-q)pR^4 \sum_{k=1}^{i} (pqR^4)^k \right)$$

for $i \geq 1$. Then, we have

$$pt(e_{2i}) + (1-p)t(\overline{e_{2i-1}}) = \frac{t(e_{2i-1})}{R},\ qt(e_{2i+1}) + (1-q)t(\overline{e_{2i}}) = \frac{t(e_{2i})}{R} \text{ and } t(\overline{e_{i-1}}) = \frac{t(\overline{e_i})}{R},$$

so Γ has property WSG with (t, B, R^{-1}).

Generalizing geometrically finite groups, we consider $\Gamma < \mathrm{Aut}(\mathcal{T})$ for which a quotient graph of groups $\Gamma\backslash\backslash\mathcal{T}$ is a union of a finite graph of groups, finite rays and finite funnels. Consider the following ray (Figure 4.3). Let e_i be the edge with $\partial_0 e_i = a_{i-1}$ and $\partial_1 e_i = a_i$. Then,

$$p_{e_i e_{i+1}} = \frac{|\Gamma_{\xi_i}| \nu^+_{a_i}(\mathcal{O}_{e_{i+1}})}{|\Gamma_{\xi_i,\xi_{i+1}}| \nu^+_{a_i}(\mathcal{O}_{e_i})} = \frac{s_i \cdot \nu^+_{a_i}(\mathcal{O}_{e_{i+1}})}{\nu^+_{a_i}(\mathcal{O}_{e_i})}.$$

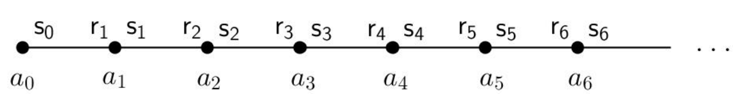

Figure 4.3: General ray

Example 4.2.10. First, consider the case $s_i = 2$ and $r_i = q - 1$. Let $\Gamma < \mathrm{Aut}(\mathcal{T})$ be a full discrete subgroup such that the decomposition of quotient graph of groups $\Gamma\backslash\backslash\mathcal{T}$ consists of a finite graph of groups D, finite funnels, and finite rays of type $(2, q-1)$. This is a quotient of a $(q+1)$-regular tree, but not geometrically finite. There is $\alpha > 0$ such that

$$\alpha < p_{e_i,e_{i+1}} < 1-\alpha \text{ and } \alpha < p_{\overline{e_{i+1}},\overline{e_i}} < 1-\alpha.$$

Choose a sequence $t_i \geq 1$ satisfying the following conditions: there exists $0 < \rho < 1$ such that for any $i = 0, 1, 2, \cdots$,

(1) $p_{e_i,e_{i+1}} t_{i+1} + (1 - p_{e_i,e_{i+1}}) t_{i-1} \leq t_i \rho$; and

(2) $p_{\overline{e_{i+1}},\overline{e_i}} t_{i-1} + (1 - p_{\overline{e_{i+1}},\overline{e_i}}) t_{i+1} \leq t_i \rho$.

Then, Γ has property WSG with (t, ED, ρ).

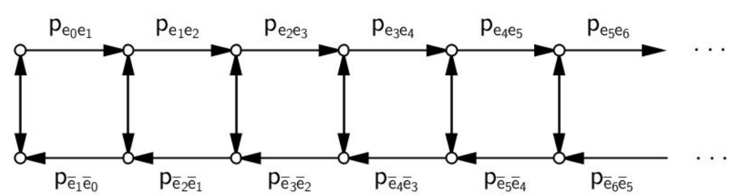

Figure 4.4: Markov chain associated to a ray

Example 4.2.11. Now we consider the case that the ray of groups itself is not even an *expander diagram*. The most famous example is appeared in [BeLu]. In this example, for each i, either $r_i = 1$ or q and $s_i = q + 1 - r_i$. Although the regular representation of $\mathrm{Aut}(\mathcal{T})$ into $L^2(\Gamma\backslash\mathrm{Aut}(\mathcal{T}))$ has no spectral gap (see [BeLu]), we have the exponential mixing property of the geodesic 2-translation map $\phi^{\circ 2}$. Indeed, as in the previous example, we can choose a sequence t_i for $i \geq 0$ such that

(1) $p_{e_i,e_{i+1}} t_{i+1} + (1 - p_{e_i,e_{i+1}}) t_{i-1} \leq t_i \rho$; and

(2) $p_{\overline{e_{i+1}},\overline{e_i}} t_{i-1} + (1 - p_{\overline{e_{i+1}},\overline{e_i}}) t_{i+1} \leq t_i \rho$.

Now we can apply Lemma (5.2.4) and Proposition (4.2.5).

4.2.3 Equidistribution and counting discrete points

Using arguments of [PPS] and [BPP], one can obtain error rate on the number of edge paths of length at most n in $\Gamma\backslash\backslash\mathcal{T}$ with weights $e^{\int_x^{\gamma y} \widetilde{F}}$ using the equidistribution of the skinning measure $d\sigma_{\widetilde{\mathcal{H}}}$. The following corollary gives the precise statement.

Corollary 4.2.12. *Let x be a degree q^d+1 vertex of the Bruhat-Tits tree $\mathcal{T}$ of G and suppose that $(\Gamma, \widetilde{F})$ is a pair of discrete subgroup $\Gamma < G$ and a potential $\widetilde{F}$ for Γ with property WSG. Let $\mathcal{N}_x(n)$ be the weighted counting function of the closed path in $\Gamma\backslash\backslash\mathcal{T}$ of length at most n with base point Γx. In other words, let*

$$\mathcal{N}_x(n) = \sum_{\gamma:\, d_{\mathcal{T}}(\gamma x, x) \leq n} e^{\int_x^{\gamma x} \widetilde{F}}.$$

Then as $n \to \infty$, we have

$$\mathcal{N}_x(2n) = \frac{e^{2\delta_{\Gamma,F}} \|\nu_x^-\| \|\nu_x^+\| |\Gamma_x|}{(e^{2\delta_{\Gamma,F}} - 1) \|m_{\Gamma,F}^{\nu^-,\nu^+}\|} e^{2n\delta_{\Gamma,F}} + O(e^{(2\delta_{\Gamma,F} - \kappa)n})$$

for some $\kappa > 0$.

Chapter 5

Pointwise equidistribution with an error rate in the space of unimodular lattices

As $\mathbb{Z}$ has many properties in common with $\mathbb{F}_q[t]$, the ring of polynomials over a finite field, it is suitable to relate the arithmetic of their fields of quotients $\mathbb{Q}$ and $\mathbb{F}_q(t^{-1})$. In particular, we study Diophantine approximation in function fields, namely, a quantitative analysis of the density of $\mathbb{F}_q(t)$ in $\mathbb{F}_q((t^{-1}))$.

In this chapter we focus on the space of unimodular $\mathbf{Z}$-lattices in $\mathbf{K}^n$ for $\mathbf{K} = \mathbb{F}_q((t^{-1}))$ and $\mathbf{Z} = \mathbb{F}_q[t]$. We investigate the dynamical properties of diagonal actions on the measure space $(SL(d, \mathbf{K})/SL(d, \mathbf{Z}), \mu)$ for the $SL(d, \mathbf{K})$-invariant probability measure on μ. We also apply our results to find the asymptotic of the number of solutions of certain Diophantine inequalities in $\mathbf{K}^d$ with weights and directions. These are the function fields analogues of the papers [KSW] and [Shi] which dealt with the action of the diagonal elements in expanding cones in real Lie groups.

5.1 Uniform non-divergence

The norm $|\cdot|$ on $\mathbf{K}$ is given by $q^{\deg(\cdot)}$. For a vector $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbf{K}^n$, we define $|\mathbf{x}| = \max\limits_{1\le i\le d}|x_i|$. Since $SL(d,\mathbf{K})$ acts transitively on the space Ω of unimodular free $\mathbf{Z}$-module of rank d in $\mathbf{K}^n$ and the stabilizer of $\mathbf{Z}^d$ is $SL(d,\mathbf{Z})$, we can identify $X = SL(d,\mathbf{K})/SL(d,\mathbf{Z})$ with Ω.

For each $\Lambda \in \Omega$, we can choose a basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_d$ and define

$$\|\Lambda\| = |\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_d|.$$

Also, let us define a function $\delta\colon \Omega \to \mathbb{R}^+$ by

$$\delta(\Lambda) = \inf_{\mathbf{v}\in\Lambda\setminus\{0\}} |\mathbf{v}|.$$

If $\delta(g\Gamma) < \epsilon$, then there exists a non-zero vector $\mathbf{w} \in \mathbf{Z}^n$ such that $|g\mathbf{w}| < \epsilon$.

Let us denote by $\mathfrak{a}^+$ the set of d-tuples $\mathbf{a} = (a_1, \cdots, a_d) \in \mathbb{N}^d$ such that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} a_{m+j}.$$

For $\mathbf{a} = (a_1, \cdots, a_d) \in \mathfrak{a}^+$ and $\mathbf{y} \in \mathrm{Mat}_{m\times n}(\mathbf{K})$, let us define

$$g_{\mathbf{a}} = \mathrm{diag}(t^{a_1}, \cdots, t^{a_m}, t^{-a_{m+1}}, \cdots, t^{-a_{m+n}}) \quad \text{and} \quad u_{\mathbf{y}} = \begin{pmatrix} I_m & \mathbf{y} \\ 0 & I_n \end{pmatrix}.$$

For the sake of brevity, let $M = \mathrm{Mat}_{m\times n}(\mathbf{K})$ and $H = \{u_{\mathbf{y}} \,|\, \mathbf{y} \in M\}$.

5.1.1 Good functions

Suppose we are given a metric space X with a locally finite Borel measure μ on X. Let C and α be positive numbers. For any locally compact field k we say a function $f\colon V \subset X \to k$ is (C,α)-*good* on V with respect to μ if for any open ball $B \subset V$ and for any $\epsilon > 0$, we have

$$\mu(\{\mathbf{v} \in B \,|\, |f(\mathbf{v})| < \epsilon\}| \leq C \left(\frac{\epsilon}{\sup_B |f|}\right)^{\alpha} \mu(B).$$

The following lemma is proved in the Lemma 4.1 of [To] for $d = 1$ and in the Lemma 2.4 of [KT] for $d \geq 2$.

Lemma 5.1.1. *For any $r, s \in \mathbb{N}$, there exist a constant C, depending only on r and s, such that every polynomial $f \in \mathbf{K}[x_1, \cdots, x_r]$ of degree less than or equal to s is $(C, 1/rs)$-good on $\mathbf{K}^d$ with respect to λ, where λ is the Haar measure on $\mathbf{K}^d$.*

5.1.2 Quantitative non-divergence

Let $Q_\epsilon = \{\Lambda \in \Omega \,|\, \delta(\Lambda) \geq \epsilon\}$. By Mahler's compactness criterion, it follows that Q_ϵ is compact. We say a $\mathbf{Z}$-submodule Δ of $\mathbf{Z}^d$ is *primitive* if $\Delta = \mathbf{K}\Delta \cap \mathbf{Z}^d$.

Theorem 5.1.2 ([Gh], Theorem 4.5)**.** *Let $d, l \in \mathbb{N}, C, \alpha > 0$ and $0 < \rho < 1$ be given. Let a ball $B \subset \mathbf{K}^l$ and a continuous map $h\colon B \to GL(d, \mathbf{K})$ be given. For any nonzero primitive submodule Δ of $\mathbf{Z}^d$, let $\psi_\Delta(\mathbf{y}) = \|h(\mathbf{y})\Delta\|$. Assume that*

(1) The function $\psi_\Delta\colon B \to \mathbb{R}$ is (C,α)-good on B.

(2) $\sup_B \psi_\Delta \geq \rho$.

(3) For each $\mathbf{y} \in B$, there are only finitely many Δ for which $\psi_\Delta(\mathbf{y}) < \rho$.

Then for any $0 < \epsilon < \rho$, we have

$$\lambda(\{\mathbf{y} \in B \,|\, \delta(h(\mathbf{y})\mathbf{Z}^d) < \epsilon\}) \leq dC\left(\frac{\epsilon}{\rho}\right)^\alpha \lambda(B).$$

To use the above theorem for $h(\mathbf{y}) = g_\mathbf{a} u_\mathbf{y}$, we need to check the condition (2):

Lemma 5.1.3. *There exists $\beta > 0$ with the following property. Let B be a cube centered at e in M. Then there is $b > 0$ such that for each $r = 1, \cdots, d-1$, $\mathbf{v} \in \bigwedge^r \mathbf{K}^d$ and $\mathbf{a} \in \mathfrak{a}^+$, one has*

$$\sup_{\mathbf{y} \in B} |g_\mathbf{a} u_\mathbf{y} \mathbf{v}| \geq b q^{\beta \lfloor \mathbf{a} \rfloor} |\mathbf{v}|.$$

Proof. We follow the idea of the proof for the real Lie groups appeared in [Sha]. Let $V = \bigwedge^r \mathbf{K}^d$ and $\rho\colon G \to GL(V)$ be the canonical representation. Let $W = \{\mathbf{v} \in V \colon \rho(u_\mathbf{y})\mathbf{v} = \mathbf{v} \text{ for all } \mathbf{y} \in M\}$ and denote by p_W the projection $V \to W$ of the vector space. The matrix of $\rho(u_\mathbf{y})$ is upper-triangular and the dimension of W is the maximum size of the identity submatrix of $\rho(u_\mathbf{y})$ obtained by deleting some collections of rows and columns.

Let $\{\mathbf{f}_1, \cdots \mathbf{f}_{mn}\}$ be the standard basis of M. Then $(\rho(u_{\mathbf{f}_k}) - Id)^2 = 0$ for each $k = 1, \cdots, mn$. Let $\mathcal{I} = \{(i_1, \cdots, i_{mn}) \colon 0 \leq i_k \leq 1, k = 1, \cdots, mn\}$.

For each $I = (i_1, \cdots, i_{mn})$ and

$$\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_{n+1} & y_{n+2} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{(m-1)n+1} & y_{(m-1)n+2} & \cdots & y_{mn} \end{pmatrix},$$

let us define

$$\tau(u^I_{\mathbf{y}}) = (\rho(u_{y_1\mathbf{f}_1}) - \mathrm{Id})^{i_1} \cdots (\rho(u_{y_{mn}\mathbf{f}_{mn}}) - \mathrm{Id})^{i_{mn}}.$$

Given $\mathbf{f} = \mathbf{f}_1 + \cdots + \mathbf{f}_{mn}$, let $T\colon V \to \bigoplus\limits_{I\in\mathcal{I}} W$ (the 2^{mn}-copies of W) be the linear transformation defined by

$$T(\mathbf{v}) = [p_W(\tau(u^I_{\mathbf{f}})\mathbf{v})]_{I\in\mathcal{I}}.$$

Since the matrix representing $\rho(u_{\mathbf{y}})$ is upper-triangular, for each vector $\mathbf{v}$ in V there exists $I \in \mathcal{I}$ such that $p_W(\tau(u^I_{\mathbf{f}})\mathbf{v})$ is non-zero. It follows that T is injective.

For each k, fix $x_{k,0}$ and $x_{k,1}$ in $\mathbf{K}$ such that $x_{k,i}\mathbf{f}_k \in B$ and $|x_{k,1}| \neq |x_{k,2}|$. Take $\mathbf{x}_I = \sum_{k=1}^{mn} x_{k,i_k}\mathbf{f}_k$ and let us define

$$S_{\mathbf{x}_I}(\mathbf{v}) = [p_W(\rho(u_{\mathbf{x}_I})\mathbf{v})]_{I\in\mathcal{I}}.$$

Then, it follows that $S_{\mathbf{x}_I}(\mathbf{v}) = M_{\mathbf{x}_I} \circ T(v)$ with the invertible linear transformation $M_{\mathbf{x}_I}\colon \bigoplus\limits_{I\in\mathcal{I}} W \to \bigoplus\limits_{I\in\mathcal{I}} W$ satisfying $|\det(M_{\mathbf{x}_I})| = \prod_{k=1}^{mn} |(x_{k,0} - x_{k,1})|^2$.

Therefore, there exist $c_0, c_1, c_2 > 0$ such that

$$q^{\lfloor \mathbf{a} \rfloor}|\mathbf{v}| \le c_0 q^{\lfloor \mathbf{a} \rfloor}|S_{\mathbf{x}_I}(\mathbf{v})| \le c_1 q^{\lfloor \mathbf{a} \rfloor} \sup_{\mathbf{y}\in B} |p_W(\rho(u_{\mathbf{y}})\mathbf{v})|$$
$$\le c_2 \sup_{\mathbf{y}\in B} |p_W(\rho(g_{\mathbf{a}}u_{\mathbf{y}})\mathbf{v})| \le c_2 \sup_{\mathbf{y}\in B} |\rho(g_{\mathbf{a}}u_{\mathbf{y}})\mathbf{v}|$$

which completes the proof. □

Example 5.1.4. Let $G = SL(3, \mathbf{K})$ with $m = 1$ and $n = 2$, $V = \bigwedge^2 \mathbf{K}^4$ and $\rho\colon G \to GL(V)$. In this case, $W = \mathbf{K}\vec{e_1} \wedge \vec{e_2} + \mathbf{K}\vec{e_1} \wedge \vec{e_3}$. For

$$u_{\mathbf{y}} = \begin{pmatrix} 1 & y_1 & y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\rho(u_{\mathbf{y}})$ is given by

$$\begin{pmatrix} 1 & 0 & -y_2 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

For a vector $\mathbf{v}$ in V given by $a\vec{e_1} \wedge \vec{e_2} + b\vec{e_1} \wedge \vec{e_3} + c\vec{e_2} \wedge \vec{e_3}$, we have

$$T(\mathbf{v}) = \begin{bmatrix} a\vec{e_1} \wedge \vec{e_2} + b\vec{e_1} \wedge \vec{e_3} \\ c\vec{e_1} \wedge \vec{e_3} \\ -c\vec{e_1} \wedge \vec{e_2} \\ 0 \end{bmatrix}$$

and

$$S(\mathbf{v}) = \begin{bmatrix} (a - x_{2,0}c)\vec{e_1} \wedge \vec{e_2} + (b + x_{1,0}c)\vec{e_1} \wedge \vec{e_3} \\ (a - x_{2,0}c)\vec{e_1} \wedge \vec{e_2} + (b + x_{1,1}c)\vec{e_1} \wedge \vec{e_3} \\ (a - x_{2,1}c)\vec{e_1} \wedge \vec{e_2} + (b + x_{1,0}c)\vec{e_1} \wedge \vec{e_3} \\ (a - x_{2,1}c)\vec{e_1} \wedge \vec{e_2} + (b + x_{1,1}c)\vec{e_1} \wedge \vec{e_3} \end{bmatrix}.$$

Then $S(\mathbf{v}) = M \circ T(\mathbf{v})$ with the invertible linear transformation $M \colon \bigoplus_{I \in \mathcal{I}} W \to \bigoplus_{I \in \mathcal{I}} W$ given by

$$\begin{pmatrix} 1 & x_{1,0} & x_{2,0} & x_{1,0}x_{2,0} \\ 1 & x_{1,1} & x_{2,0} & x_{1,1}x_{2,0} \\ 1 & x_{1,0} & x_{2,1} & x_{1,0}x_{2,1} \\ 1 & x_{1,1} & x_{2,1} & x_{1,1}x_{2,1} \end{pmatrix}.$$

In this case, $|\det(M)| = |(x_{1,0} - x_{1,1})|^2 |(x_{2,0} - x_{2,1})|^2$.

Corollary 5.1.5. *Let $L \subset X$ be compact and let $B \subset H$ be a ball centered at $e \subset H$. Then there exists $N = N(B, L)$ such that for every $0 < \epsilon < 1$, any $z \in L$ and any $\mathbf{a} \in \mathfrak{a}^+$ with $\lfloor \mathbf{a} \rfloor \geq N$ we have*

$$\lambda(\{\mathbf{y} \in B \,|\, \delta(g_{\mathbf{a}} u_{\mathbf{y}} z) < \epsilon\}) \ll \epsilon^{1/mn(m+n-1)} \lambda(B).$$

Proof. Let $h \colon M \to G$ be given by $h(\mathbf{y}) = g_{\mathbf{a}} u_{\mathbf{y}}$. Then $\mathbf{y} \mapsto \|h(\mathbf{y})\Gamma\|$ is (C, α)-good, due to the Lemma (5.1.1). It follows from the compactness of L and discreteness of $\bigwedge^r \mathbf{Z}^d$ in $\bigwedge^r \mathbf{K}^d$ that

$$\inf\{|s\mathbf{v}| \colon s\Gamma \in L, \mathbf{v} \in \bigwedge^r \mathbf{Z}^d - \{0\}, r = 1, \cdots, d-1\}$$

is positive. Together with Lemma (5.1.3), we can find $c > 0$ such that for any $s \in \pi^{-1}(L)$ we have $\sup_{\mathbf{y}\in B} |g_{\mathbf{a}} u_{\mathbf{y}} s\mathbf{v}| \geq c q^{\lfloor \mathbf{a} \rfloor}$. In view of the Theorem (5.1.2), the conditions are satisfied with $\rho = 1$. □

5.2 Pointwise equidistribution

This section is devoted to the prove of pointwise equidistribution theorem for compactly supported functions. The first key result we use is the exponential mixing property of the regular action of $g_{\mathbf{a}}$ on $L^2(X)$.

Proposition 5.2.1 (cf. [AGP], Theorem 2.1)**.** *Given any functions ϕ and ψ in $L^2(G/\Gamma)$, there exist C_0 and $\delta_0 > 0$ such that*

$$\left| \int_{G/\Gamma} \phi(x)\psi(g_{\mathbf{a}}x)d\mu(x) - \int_{G/\Gamma} \phi d\mu \int_{G/\Gamma} \psi d\mu \right| \leq C_0 \|\phi\|_2 \|\psi\|_2 e^{-\delta_0 \lfloor \mathbf{a} \rfloor}.$$

We denote by $\lfloor \mathbf{a} \rfloor$ the value of $\min\{a_i \,|\, i = 1, \cdots, m+n\}$. The following proposition says that the $g_{\mathbf{a}}$-translates of H-orbits become effectively equidistributed. For the case of *equal weights* which we refer to the case

$$\mathbf{a} = (n, \cdots, n, m, \cdots, m),$$

the proof is presented in the Proposition 2.4 of [Moh].

Proposition 5.2.2 (Effective equidistribution)**.** *There exists $\delta_1 > 0$ such that the following holds. For any $f \in C_c(M)$, $\phi \in C_c(X)$ and for any compact $L \subset X$, there exists $C_1 = C_1(f, \phi, L)$ such that for all $\Lambda \in L$ and $n \geq 0$, we* *have*

$$\left|\int_M f(\mathbf{x})\phi(g_{\mathbf{a}}u_{\mathbf{x}}\Lambda)dm(\mathbf{x}) - \int_M f(\mathbf{x})dm(\mathbf{x})\int_X \phi(x)d\mu(x)\right| \leq C_1 e^{-\delta_1\lfloor\mathbf{a}\rfloor}.$$

Proof. We follow the proof of Theorem 1.3 in [KM]. Given any $\epsilon > 0$, let

$$A = \{\mathbf{y} \in B \mid \delta(g_{\mathbf{a}}u_{\mathbf{y}}\Gamma) < \epsilon\}$$

and let $B = M\backslash A$. Without loss of generality, we may assume that $\int_X \phi\, d\mu = 0$. Take a function $\varphi \in C_c(M)$ with $\int_M \varphi dm = 1$. Let $\mathbf{s} = (\lfloor\mathbf{a}\rfloor - 1)(1, \cdots, 1)$ and let $\mathbf{t} = \mathbf{a} - \mathbf{s}$. It follows that $\mathbf{t} \in \mathfrak{a}^+$. Let $\mathbf{z} \in M$ be the element for which $u_{\mathbf{z}} = g_{\mathbf{t}}^{-1}u_{\mathbf{y}}g_{\mathbf{t}}$. We have

$$\begin{aligned}
&\int_A f(\mathbf{x})\phi(g_{\mathbf{a}}u_{\mathbf{x}}\Lambda)dm(\mathbf{x}) = \int_A f(\mathbf{x})\phi(g_{\mathbf{a}}u_{\mathbf{x}}\Lambda)dm(\mathbf{x})\int_M \varphi(\mathbf{y})dm(\mathbf{y})\\
=&\int_A\int_M f(\mathbf{z}\mathbf{x})\varphi(\mathbf{y})\phi(g_{\mathbf{s}}u_{\mathbf{y}}g_{\mathbf{t}}u_{\mathbf{x}}\Lambda)dm(\mathbf{y})dm(\mathbf{x})\\
\ll&\lambda(A)\|f\|_\infty\|\phi\|_\infty.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
&\left|\int_B f(\mathbf{x})\phi(g_{\mathbf{a}}u_{\mathbf{x}}\Lambda)dm(\mathbf{x})\right| = \left|\int_B f(\mathbf{x})\phi(g_{\mathbf{a}}u_{\mathbf{x}}\Lambda)dm(\mathbf{x})\int_M \varphi(\mathbf{y})dm(\mathbf{y})\right|\\
=&\left|\int_B\int_M f(\mathbf{z}\mathbf{x})\varphi(\mathbf{y})\phi(g_{\mathbf{s}}u_{\mathbf{y}}g_{\mathbf{t}}u_{\mathbf{x}}\Lambda)dm(\mathbf{y})dm(\mathbf{x})\right|\\
\leq&\int_B\left|\int_M f(\mathbf{z}\mathbf{x})\varphi(\mathbf{y})\phi(g_{\mathbf{s}}u_{\mathbf{y}}g_{\mathbf{t}}u_{\mathbf{s}}\Lambda)dm\right|dm\\
=&\int_B\left|\int_M f_{\mathbf{x}}(\mathbf{y})\phi(g_{\mathbf{s}}u_{\mathbf{y}}g_{\mathbf{t}}u_{\mathbf{s}}\Lambda)dm\right|dm
\end{aligned}$$

$$\ll \lambda(B)e^{-\delta_1\lfloor \mathbf{s} \rfloor}.$$

by the Proposition 2.4 of [Moh]. Taking suitable φ whose support is small, we prove the result. □

Let us define

$$I^{(n,m)}_{f,\phi,\psi}(g_{\mathbf{a}}, x_1, x_2) = \int_H f(h)\phi(g^n_{\mathbf{a}} h x_1)\psi(g^m_{\mathbf{a}} h x_2) d\theta(h)$$

for $f \in C_c(H)$, $\phi, \psi \in C_c(X)$, $x_1, x_2 \in X$ and $n, m \in \mathbb{N}$. We again follow the idea of [KSW] using the double equidistribution to prove pointwise equidistribution.

Proposition 5.2.3 (Double equidistribution)**.** *There exists $\delta_2 > 0$ such that the following holds. Given $f \in C_c(H)$, $\phi, \psi \in C_c(X)$ and a compact subset L of X, there exists $C_2 = C_2(f, \phi, \psi, L)$ such that for any $x_1, x_2 \in L$ and $n, m \in \mathbb{N}$, we have*

$$\left| I^{(n,m)}_{f,\phi,\psi}(g_{\mathbf{a}}, x_1, x_2) - \int_H f d\theta(h) \int_X \phi d\mu \int_X \psi d\mu \right| \leq C_2 e^{-\delta_2 \min(n, m-n)\lfloor \mathbf{a} \rfloor}.$$

Proof. It is enough to show the statement for characteristic functions f, ϕ and ψ. There are compact open subgroups H_f of H and K_ϕ of G which leave f and ϕ invariant, respectively. Let us denote by K^l the l-th congruence subgroup of K and let $q \in C_c(H)$ be such that $q \geq 0$, $\int_H q(y)dy = 1$ and $\operatorname{supp}(q) \subset K^l$. By taking l large, we may assume that $K^l \cap H \subset H_f$ and $K^l \subset K_\phi$. Let

$y_1 = g_{\mathbf{a}}^{-n} y g_{\mathbf{a}}^{n}$. Making a change of variable $h \mapsto y_1 h$ gives us

$$I_{f,\phi,\psi}^{(n,m)} = \int_H \int_H f(y_1 h)\phi(g_{\mathbf{a}}^n y_1 h x_1)\psi(g_{\mathbf{a}}^m y_1 h x_2)k(y)dhdy.$$

Now we have

$$\left| I_{f,\phi,\psi}^{(n,m)} - \int_H \int_H f(h)\phi(g_{\mathbf{a}}^n h x_1)\psi(g_{\mathbf{a}}^m y_1 h x_2)k(y)dhdy \right| = 0$$

since $y_1 \in H_f$ and $y \in K_\phi$. Also,

$$\begin{aligned} &\left| \int_H \psi(g_{\mathbf{a}}^m y_1 h x_2)k(y)dy - \int_X \psi(x_2)d\mu \right| \\ = &\left| \int_H \psi(g_{\mathbf{a}}^{m-n} y g_{\mathbf{a}}^n h x_2)k(y)dy - \int_X \psi(x_2)d\mu \right| \le C_1 e^{-\delta_1 (m-n)\lfloor \mathbf{a} \rfloor} \end{aligned}$$

Finally,

$$\left| \int_H f(h)\phi(g_{\mathbf{a}}^n h x_1)dh - \int_H f \int_X \phi \right| \le C_1 e^{-\delta_1 n \lfloor \mathbf{a} \rfloor}.$$

This completes the proof. □

Lemma 5.2.4. *Given a probability space (Y, ν), suppose that $F\colon Y \times \mathbb{Z}_{\ge 0} \to \mathbb{R}$ satisfies*

$$\left| \int_Y F(y,n)F(y,m)d\nu \right| \le C q^{-\delta \min(n, m-n)}.$$

Then we have

$$\int_Y \left(\sum_{n=b}^{c-1} F(y,n) \right)^2 d\nu(y) \le \frac{4C(c-b)}{1-e^{-\delta}}.$$

Proof. ([KSW], Lemma 3.2)

$$\begin{aligned}
\int_(\left(\sum_{n=b}^{c-1} F(y,n) \right)^2 d\nu(y) &= \sum_{n=b}^{c-1} \sum_{m=b}^{c-1} \int_Y F(y,n)F(y,m) d\nu(y) \\
&\leq 2 \sum_{n=b}^{c-1} \sum_{m=n}^{c-1} \left| \int_Y F(y,n)F(y,m) d\nu(y) \right| \\
&\leq 2C \sum_{n=b}^{c-1} \sum_{m=n}^{c-1} \left(q^{-\delta(m-n)} + q^{-\delta n} \right) \\
&\leq 2C \sum_{n=b}^{c-1} \left(\frac{1}{1-q^{-\delta}} + q^{-\delta n}(c-n) \right) \\
&\leq \frac{4C(c-b)}{1-q^{-\delta}}.
\end{aligned}$$

□

In particular, if L_s is the set of intervals of the form $[2^i j, 2^i(j+1)) \cap \mathbb{Z}_{\geq 0}$, then we have

$$\begin{aligned}
\sum_{[b,c)\in L_s} \int_Y \left(\sum_{n=b}^{c-1} F(y,n) \right)^2 d\nu(y) &\leq \sum_{i=0}^{s-1} \sum_{j=0}^{2^{s-i}-1} \int_Y \left(\sum_{n=2^i j}^{2^i(j+1)-1} F(y,n) \right)^2 d\nu(y) \\
&\leq \sum_{i=0}^{s-1} \sum_{j=0}^{2^{s-i}-1} \frac{4C2^i}{1-q^{-\delta}} \\
&= \frac{4Cs2^s}{1-q^{-\delta}}.
\end{aligned}$$

For any given $\epsilon > 0$, let

$$Y_s = \left\{ y \in Y | \sum_{I \in L_s} \left(\sum_{n \in I} F(y,n) \right)^2 > 2^s s^{2+2\epsilon} \right\}.$$

Then,

$$\nu(Y_s) \leq \frac{4Cs^{-(1+2\epsilon)}}{1-q^{-\delta}}$$

and moreover

$$\sum_{n=0}^{k-1} F(y,n) \leq 2^{s/2} s^{3/2+\epsilon}$$

for $y \notin Y_s$ and for all $1 \leq k \leq 2^s$. By Borel-Cantelli lemma, we get a measurable subset Z_ϵ of Y with full measure such that for every $y \in Z_\epsilon$ there exists $s_y \in \mathbb{N}$ such that $y \notin Y_s$ whenever $s \geq s_y$.

Theorem 5.2.5 (Pointwise equidistribution)**.** *For any given $x \in X$, $\phi \in C_c(X)$ and $\epsilon > 0$, we have*

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi(g_{\mathbf{a}}^n hx) = \int_X \phi d\mu + O\left(N^{-1/2}(\log N)^{\frac{3}{2}+\epsilon}\right)$$

for almost every $h \in H$.

Proof. ([KSW], Lemma 3.4 and Lemma 3.5) Given $x \in X$, $\phi \in C_c(X)$ and $\epsilon > 0$, let ν be the probability measure on X defined by

$$\int \psi d\nu = \int_M f(\mathbf{y})\psi(u_{\mathbf{y}}x)dm(\mathbf{y}).$$

Let $\alpha = \int_X \varphi d\mu$. Then there exist $C > 0$ and $\delta > 0$ such that

$$\left|\int_X (\varphi(g_{\mathbf{a}}^n x) - \alpha)(\varphi(g_{\mathbf{a}}^n x) - \alpha)d\nu\right| \leq Cq^{-\delta \min(n, m-n)}$$

holds for every $m \geq n \geq 0$. Applying the argument below Lemma 5.2.4 to $F(x,n) = \phi(g_{\mathbf{a}}^n x) - \alpha$, we obtain a full measure subset Z_ϵ of Y with the following property. For every $y \in Z_\epsilon$, we have

$$\left|\sum_{n=0}^{N-1} F(y,n)\right| \leq 2^{s/2}s^{3/2+\epsilon} \leq (2N)^{1/2}\log^{3/2+\epsilon}(2N)$$

when $N \geq 2^{s_y-1}$. This enables us to reach the conclusion. □

5.3 Diophantine approximation with weights

Due to [DKL], we have a quantitative Khintchine-Groshev type theorem over a field of formal series as follows. Let $V = \{q^{-n} : n \in \mathbb{N}\}$ and $\psi\colon \mathbb{R}^+ \to V$. For $\mathbf{p} \in \mathbf{K}^n$ and $\mathbf{q} \in \mathbf{K}^m$, consider the inequality

$$\|\mathbf{q}A - \mathbf{p}\|_\infty < \psi(\|\mathbf{q}\|_\infty). \tag{5.3.1}$$

Let $\epsilon > 0$ be arbitrary and let

$$\Phi(Q) = m(q-1)q^{m-1}\sum_{r=0}^{Q} q^{rm}\psi(q^r)^n.$$

If we denote by $N(Q,A)$ the number of solutions to (5.3.1), then we have

$$N(Q,A) = \Phi(Q) + O(\Phi(Q)^{1/2}\log^{3/2+\epsilon}(\Phi(Q)))$$

for almost every $A \in M$.

In this section we consider the Diophantine inequalities concerning the weighted quasi-norms given by

$$|\mathbf{x}|_\alpha = \max_{1\le i\le m} |x_i|^{1/a_i} \qquad \text{and} \qquad |\mathbf{y}|_\beta = \max_{m+1\le j\le d} |y_j|^{1/a_j}.$$

In this case, the set $E_{T,R}$ given by

$$\left\{(\mathbf{x},\mathbf{y}) \in \mathbf{K}^m \times \mathbf{K}^n \colon |\mathbf{x}|_\alpha < \frac{q^R}{|\mathbf{y}|_\beta}, 1 \le |\mathbf{y}|_\beta < q^T\right\}$$

is invariant under the left action of the group $SL(d,\mathcal{O})$. The volume of $E_{T,R}$ is

$$\begin{aligned}\lambda(E_{T,R}) &= \sum_{k\in\mathbb{Q},0\le k\le T} \lambda(\{|\mathbf{x}|_\alpha < q^{R-k}\})\lambda(\{|\mathbf{y}|_\beta = q^k\}) \\ &= \sum_{k\in\mathbb{Q},0\le k\le T} (q^{\lfloor (R-k)a_1\rfloor+\cdots\lfloor (R-k)a_m\rfloor+m})(q^{\lfloor ka_{m+1}\rfloor+\cdots\lfloor ka_{m+n}\rfloor+n} - q^{\lceil ka_{m+1}\rceil+\cdots\lceil ka_{m+n}\rceil}).\end{aligned}$$

Since the above is a finite sum, it is well-defined. It also satisfies the homogeneity with respect to integers: $\lambda(E_{T,R}) = T\lambda(E_{1,R})$ for all $T \in \mathbb{N}$.

Let us also define

$$F_{S,R} = \left\{(\mathbf{x},\mathbf{y}) \in \mathbf{K}^m \times \mathbf{K}^n \colon |\mathbf{x}|_\alpha < \frac{q^R}{|\mathbf{y}|_\beta}, 1 \le |\mathbf{x}|_\alpha < q^S\right\}.$$

There is a one-to-one correspondence between the set of nonzero solutions of

$$|A\mathbf{q} - \mathbf{p}|_\alpha < \frac{q^R}{|\mathbf{q}|_\beta}, \qquad 1 \le |\mathbf{q}|_\beta < q^T \tag{5.3.2}$$

and the intersection of $u_A\Lambda_0$ with the set $E_{T,R}$. Modifying the argument of [DKL], we can compute the number $\widetilde{N}_R(T,A)$ of solutions satisfying (5.3.2).

Theorem 5.3.3. *Let $\Psi_R(T) = \lambda(E_{T,R})$. Then, we have*

$$\widetilde{N}_R(T,A) = \Psi_R(T) + O\left(\Psi_R(T)^{\frac{1}{2}} \log^{2+\epsilon}(\Psi_R(T))\right)$$

for almost every $A \in M$.

Proof. Let us define

$$B_{\mathbf{q}} = \{A \in I^{mn} \colon \inf_{\mathbf{p}\in Z^m} |A\mathbf{q} - \mathbf{p}|_\alpha < \psi(|\mathbf{q}|_\beta)\}.$$

Then $\lambda(B_{\mathbf{q}}) = \psi(|\mathbf{q}|_\beta)^{a_1+\cdots a_m}$ and $\lambda(B_{\mathbf{q}} \cap B_{\mathbf{q}'}) = \lambda(B_{\mathbf{q}})\lambda(B_{\mathbf{q}'})$ for linearly independent vectors $\mathbf{q}, \mathbf{q}' \in Z^n$ (cf. [DKL] Proposition 3 and 4). Let $d(\mathbf{q})$ be the number of common divisors in $\mathbf{Z}$ of the coordinates of $\mathbf{q}$ and let $\tau_{\mathbf{q}} = \psi(|\mathbf{q}|_\beta)^{a_1+\cdots+a_m}$. Since we only get contributions from the elements corresponding to the pairs of parallel vectors $\mathbf{q}$ and $\mathbf{q}'$, we have

$$\int \left(\sum_{q^s < |\mathbf{q}|_\beta \le q^t} \chi_{B_{\mathbf{q}}}(A) - \sum_{q^s < |\mathbf{q}|_\beta \le q^t} \psi(|\mathbf{q}|)^{a_1+\cdots a_m} \right)^2 dA \ll \sum_{q^s < |\mathbf{q}|_\beta \le q^t} \tau_{\mathbf{q}}$$

for $s < t$. The Lemma 10 in [Sp79] implies that we have for almost every A,

$$\begin{aligned} N(T,A) &\stackrel{\text{def}}{=} \sum_{1 \le |\mathbf{q}|_\beta < q^T} \chi_{B_{\mathbf{q}}}(A) \\ &= \sum_{1 \le |\mathbf{q}|_\beta < q^T} \psi(|\mathbf{q}|_\beta)^{a_1+\cdots a_m} + O(S(T)^{1/2} \log^{3/2+\epsilon} S(T)) \end{aligned}$$

for $S(T) = \sum_{1\le |\mathbf{q}|_\beta < q^T} \tau_{\mathbf{q}}$. We immediately obtain

$$\begin{aligned}
&\sum_{1\le |\mathbf{q}|_\beta < q^T} \left(\frac{q^R}{|\mathbf{q}|_\beta}\right)^{a_1+\cdots a_m} \\
&= \sum_{k\in\mathbb{Q}, 0\le k<T} \#\{|\mathbf{q}|_\beta = q^k\} q^{(R-k)(a_1+\cdots+a_m)} \\
&= \sum_{k\in\mathbb{Q}, 0\le k<T} (q^{\lfloor ka_{m+1}\rfloor+\cdots\lfloor ka_{m+n}\rfloor+n} - q^{\lceil ka_{m+1}\rceil+\cdots\lceil ka_{m+n}\rceil}) q^{(R-k)(a_1+\cdots+a_m)}.
\end{aligned}$$

Therefore, it remains for us to show that $S(T) = O(\Psi_R(T)\log\Psi_R(T))$ when $\psi(|\mathbf{q}|_\beta) = \dfrac{q^R}{|\mathbf{q}|_\beta}$. In fact, using the Dirichlet series of the number of monic divisors ([Ros], Chapter 2, page 17), we have

$$\begin{aligned}
S(T) &= \sum_{1\le |\mathbf{q}|_\beta < q^T} \chi_{B_{\mathbf{q}}} \sum_{d|(q_1,\cdots,q_n)} 1 \\
&\ll \sum_{k\in\mathbb{Q},0\le k<T} \sum_{l=0}^{\lfloor k\rfloor} \sum_{\substack{|\mathbf{q}|_\beta = q^k \\ |GCD(q_1,\cdots,q_n)|=|L|=q^l}} \left(\frac{q^R}{|\mathbf{q}|_\beta}\right)^{a_1+\cdots+a_m} \sum_{d|L} 1 \\
&\ll \sum_{k\in\mathbb{Q},0\le k<T} q^{(R-k)(a_1+\cdots+a_m)} \sum_{l=0}^{\lfloor k\rfloor} \sum_{|\mathbf{v}|_\beta = q^{k-l}} (l+1) q^l \\
&\ll \Psi_R(T)\log\Psi_R(T).
\end{aligned}$$

This enables us to complete the proof. □

Lemma 5.3.4. *For*

$$E_{S,R} = \left\{(\mathbf{x},\mathbf{y}) \in \mathbf{K}^m \times \mathbf{K}^n \colon |\mathbf{x}|_\alpha < \frac{q^R}{|\mathbf{y}|_\beta}, 1 \le |\mathbf{y}|_\beta < q^T\right\}$$

and

$$F_{S,R} = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbf{K}^m \times \mathbf{K}^n : |\mathbf{x}|_\alpha < \frac{q^R}{|\mathbf{y}|_\beta}, 1 \leq |\mathbf{x}|_\alpha < q^S \right\}$$

defined as above of the Theorem 5.3.3, we have $\lambda(E_{S,R}) = \lambda(F_{S,R})$*.*

Proof. Since $\lambda(E_{S,R}) = \frac{S}{R}\lambda(E_{R,R})$ and $\lambda(F_{S,R}) = \frac{S}{R}\lambda(F_{R,R})$, it suffices to show that $\lambda(E_{R,R}) = \lambda(F_{R,R})$. This is equivalent to that $\lambda\{(\mathbf{x}, \mathbf{y}) \in \mathbf{K}^m \times \mathbf{K}^n : |\mathbf{x}|_\alpha < q^R, |\mathbf{y}|_\beta < 1\} = \lambda\{(\mathbf{x}, \mathbf{y}) \in \mathbf{K}^m \times \mathbf{K}^n : |\mathbf{x}|_\alpha < 1, |\mathbf{y}|_\beta < q^R\}$. Since both of these sets have measure $q^{(R+1)(a_1+\cdots a_m)}$, we get the conclusion. □

Proposition 5.3.5. *We have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \widehat{\chi}_{E_{T,R}}(g_{\mathbf{a}}^n u_A \Lambda_0) = \lambda(E_{T,R})$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \widehat{\chi}_{F_{S,R}}(g_{\mathbf{a}}^n u_A \Lambda_0) = \lambda(F_{S,R}).$$

Proof. It holds that

$$\begin{aligned} \#\left[\Lambda \cap (E_{N,R}(B,C) \backslash E_{T,R}(B,C))\right] &\leq \frac{1}{T} \sum_{n=0}^{N-1} \widehat{\chi}_{E_{T,R}(B,C)}(g_{\mathbf{a}}^n \Lambda) \\ &\leq \#\left[\Lambda \cap (E_{N+T,R}(B,C))\right] \end{aligned}$$

for any lattice Λ. Now

$$\begin{aligned} \frac{T}{N} \#\left[u_A \Lambda_0 \cap (E_{N,R} \backslash E_{T,R})\right] &\leq \frac{1}{N} \sum_{n=0}^{N-1} \widehat{\chi}_{E_{T,R}}(g_{\mathbf{a}}^n u_A \Lambda_0) \\ &\leq \frac{T}{N} \#[u_A \Lambda_0 \cap E_{N+T,R}] \end{aligned}$$

together with Theorem 5.3.3 implies the first equation. For the second equality, if we let $F = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{K}^m \times \mathbf{K}^n : |\mathbf{x}|_\alpha < q^S, |\mathbf{y}|_\beta < q^R\}$, then

$$\frac{1}{S}\sum_{n=0}^{N-1} \widehat{\chi}_{F_{S,R}}(g_{\mathbf{a}}^n u_A \Lambda_0) \leq \#[u_A\Lambda_0 \cap (E_{N+R,R})] + \#[F \cap u_A\Lambda_0].$$

Therefore, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \widehat{\chi}_{F_{S,R}}(g_{\mathbf{a}}^n u_A \Lambda_0) \leq \lambda(E_{S,R}) = \lambda(F_{S,R})$. Conversely, there is a full measure subset M' of M for which

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \varphi(g_{\mathbf{a}}^n u_A \Lambda_0) = \int_X \varphi d\mu$$

holds for all $\varphi \in C_c(X)$ and $A \in M'$. For any $\epsilon > 0$, there exists $\varphi_1 \in C_c(\mathbf{K}^d)$ such that $\chi_{F_{S,R}}(\mathbf{v}) \geq \varphi_1(\mathbf{v})$ and $\int_{\mathbf{K}^d} \varphi_1 > \lambda(F_{S,R}) - \epsilon$. Moreover, there exists $\varphi_2 \in C_c(X)$ such that $\varphi_2(\Lambda) \leq \widehat{\varphi_1}(\Lambda) \leq \widehat{\chi_{F_{S,R}}}(\Lambda)$ and

$$\int_X \varphi_2 d\mu \geq \lambda(F_{S,R}) - 2\epsilon.$$

Since ϵ was arbitrary, for almost every $A \in M$ we have

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \widehat{\chi_{F_{S,R}}}(g_{\mathbf{a}}^n u_A \Lambda_0) \geq \lambda(F_{S,R})$$

which completes the proof. $\square$

5.4 Equidistribution with respect to $C_\alpha(X)$ and counting with directions

For $f\colon \mathbf{K}^d \to \mathbb{R}$, let us define a function $\widehat{f}$ on X by

$$\widehat{f}(\Lambda) = \sum_{\mathbf{v}\in\Lambda-\{\mathbf{0}\}} f(\mathbf{v}).$$

For a generalized topological space E equipped with a measure λ on which a unimodular group G acts transitively and preserving λ, the mean value theorem of Siegel ([Si]) has been generalized in [Mor]. In our special case when $E = \mathbf{K}^d$, $G = SL(d, \mathbf{K})$ and $\Gamma = SL(d, \mathbf{Z})$, we have

$$\int_{\mathbf{K}^d} f(\mathbf{x})d\lambda = \int_{G/\Gamma} \widehat{f}(\Lambda)d\mu.$$

In order to control the rate of growth at infinity of unbounded functions on X, let us use the following notation introduced by [EMM]: For a unimodular lattice $\Lambda \in X$ and a subgroup $\Delta \leq \Lambda$ with $L = \mathbf{K}\Delta$, let us denote by $d(\Delta)$ the volume of L/Δ and

$$\alpha(\Lambda) = \max\{d(\Delta)^{-1}\colon \Delta \leq \Lambda\}.$$

Lemma 5.4.1. *For any d and all sufficiently large r there are constants c_1 and c_2 such that if $\chi_{B_{q^r}}$ is the characteristic function of the open ball of radius q^r centered at origin in $\mathbf{K}^d$, then for all $\Lambda \in X$, we have*

$$c_1\alpha(\Lambda) \leq \widehat{\chi}_{B_{q^r}}(\Lambda) \leq c_2\alpha(\Lambda).$$

Proof. Since $\widehat{\chi}_{B_{q^r}}$ and α is invariant under the left action of the group $SL(d, \mathcal{O})$, we may assume that $\Lambda = t^{a_1}\mathbf{Z}e_1 + \cdots + t^{a_d}\mathbf{Z}e_d$ with

$$a_1 \leq a_2 \leq \cdots \leq a_k \leq 0 < a_{k+1} \leq \cdots \leq a_j \leq r < a_{j+1} \leq \cdots \leq a_d$$

and

$$\sum_{i=1}^{d} a_i = 0.$$

In this case, $\widehat{\chi}_{B_{q^r}} = \#(B_{q^r} \cap \Lambda) = q^{jr-(a_1+\cdots+a_j)}$ and

$$\alpha(\Lambda) = q^{-a_1-\cdots-a_k} = q^{a_{k+1}+\cdots+a_d}.$$

Therefore, it follows that

$$q^{-dr}\alpha(\Lambda) \leq \widehat{\chi}_{B_{q^r}} \leq q^{dr}\alpha(\Lambda).$$

□

Lemma 5.4.2. *For a given $r > 0$, there exist $T, R > 0$ such that*

$$\widehat{\chi}_{B_{q^r}} \leq \widehat{\chi}_{E_{T,R}} + \widehat{\chi}_{F_{S,R}}$$

Proof. We observe that $\Lambda \cap B_{q^d} \neq \phi$ for all unimodular lattices Λ in $\mathbf{K}^d$. Moreover, for any $\mathbf{v} \in \Lambda$ we have $\#[B_{q^r} \cap \Lambda] = \#[(B_{q^r} + \mathbf{v}) \cap \Lambda]$. Meanwhile, there exist T and R such that $B_{q^r} + \mathbf{v} \subset E_{T,R} \cup F_{T,R}$ and hence $\chi_{B_{q^r}+\mathbf{v}} \leq \chi_{E_{T,R}} + \chi_{F_{T,R}}$. Therefore $\#[B_{q^r} \cap \Lambda] \leq \#[E_{T,R} \cap \Lambda] + \#[F_{T,R} \cap \Lambda]$ holds for every unimodular lattice Λ in $\mathbf{K}^d$. □

Lemma 5.4.3. *Let B_{q^r} be the open ball of radius $q^r > 0$ centered at zero in $\mathbf{K}^d$. Then we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \widehat{\chi_{B_{q^r}}}(g_{\mathbf{a}}^n u_{\mathbf{y}} \Lambda_0) = \lambda(B_{q^r})$$

for almost every $\mathbf{y} \in M$.

Proof. It follows directly from Corollary 5.4 of [KSW] and Lemma 5.4.2. □

Together with Lemma 5.4.1, this proves the following theorem.

Theorem 5.4.4 (($\{g_{\mathbf{a}}^n\}_{n\geq 1}, C_\alpha(X)$)-genericity)**.** *For every $f \in C_\alpha(X)$ and for almost every $\mathbf{y} \in M$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(g_{\mathbf{a}}^n u_{\mathbf{y}} \Lambda_0) = \int_X f d\mu.$$

Theorem 5.4.5 (Counting with directions)**.** *As $T \to \infty$,*

$$\#\{u_{\mathbf{y}}\Lambda_0 \cap E_{T,R}(A,B)\} \sim \lambda(E_{T,R}(A,B)).$$

Proof. Since $\#\{E_{T,R}(A,B)\} = \#\{E_{1,R}(A,B)\} \cdot T$, we have

$$\begin{aligned}
\#\{u_{\mathbf{y}}\Lambda_0 \cap E_{T,R}(A,B)\} &\sim \frac{1}{r} \sum_{n=0}^{T} \widehat{\chi}_{B_{q^r}}(g_{\mathbf{a}}^n u_{\mathbf{y}} \Lambda_0) \\
&\sim \frac{T}{r} \lambda(E_{r,R}(A,B)) \\
&= \lambda(E_{T,R}(A,B)).
\end{aligned}$$

This completes the proof. $\square$

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국문초록

우리는 초거리 국소체 위의 축소 가능한 군들의 구조를 공부하고 그 군들로부터 얻어지는 동질 공간의 동역학적 성질에 대하여 연구하기 위하여 유클리드 브루하-티츠 빌딩의 기하학을 사용하며 주어진 동역학계를 가산 가능한 만큼의 알파벳들로 이루어진 이동공간과 연관짓고자 한다.

먼저 우리는 차수가 1인 대수적 군의 카탄 부분군 작용이 특정한 조건 하에서 지수적으로 섞임 성질을 지닌다는 것을 증명한다. 몫공간이 기하학적으로 유한인 군이 있는 그래프인 경우에는 위의 조건을 만족하며 이 결과는 그래프 상에서 길이가 N이하인 닫힌 회로의 개수를 오차를 포함하여 세는 데에 적용할 수 있다.

또한 우리는 국소체 위의 벡터공간 안의 단위 격자들의 공간 안에서 최대 특이 반직선에 대응하는 유니포턴트 군의 임의의 궤도에 대하여 버크호프 에르고딕 정리가 거의 모든 점에서 성립한다는 사실을 증명한다. 이를 이용하여 우리는 양적 킨친-그로셰브 정리를 일반화한 정리를 보인다.

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